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Noncommutative residue and symplectic foliation

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Abstract : Let (M, ω) be a symplectic foliation with a symplectic form. Let A be an Heisenberg pseudodifferential operator. In this paper, we define the noncommutative residue of A for the symplectic foliation, using a symplectic form. Moreover, we show that is the unique trace on the algebra of Heisenberg pseudodifferential operators up to multiplication by a constant and we related it to the Dixmier trace in the given space.

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1 Introduction

The noncommutative residue of an operator is a trace on the algebra of pseudodifferential operators (ΨDO) on a compact manifold. In the context of one-dimensional integrable systems, the noncommutative residue had been studied earlier by M. Adler [1] and Y. manin [8]. To construct this residue and show its main properties, Wodzicki uses the formalism of the geometry of symplectic cones already used by Guillemin to define the symplectic residue. Wodzicki also constructs a

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morphism of complexes thanks to which the trace property of the Guillemin residue becomes evident [20] and [11]. Given a ΨDO P of order m on a compact manifold M of dimension n acting on the sections of a bundle E, the function $s \to Trace P \Delta^{-s}$ where Δ is a Laplacian on M is defined and holomorphic for $\Re s > \frac{1}{2}(m+n)$. It extends into a function which is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$ and has at most simple poles on $\mathbb{Z}$. The noncommutative residue of P is then given by :

$$resP = 2Res_{s=0} Trace P \Delta^{-s}$$

This defines a trace on the algebra $\Psi^{\mathbb{Z}}/\Psi^{-\infty}$ and it is even the only trace up to a multiplicative coefficient if M is connected and of dimension > 1. The geometry of symplectic cones also allows Wodzicki to give a local form of this residue, i.e. to construct a density $res_x(P)$ on M such that, if $f \in C_c^\infty(M)$, $res(fP) = \int_M f res_x(P)$, and such that, if (U, Φ) is a local chart, then

$$\Phi^* res_{\phi(u)}(P) = \Phi^* res_u(\Phi^* P).$$

This property implies in particular the invariance by diffeomorphism of the noncommutative residue.

The noncommutative residue has important implications in noncommutative geometry due to its links with the Dixmier trace and the local index theorem, see [6]. It has been extended to other situations such as the Boutet de Monvel algebra for the transversal elliptic calculus for foliations, see [17].

This paper is organized as follows. We start with a survey of the Heisenberg pseudodifferential operators. This is followed by an introduction to the definition and the study of certain properties of a noncommutative residue for a symplectic foliation of dimension $n = p + q$, where p is the dimension of the leaves and $q = n - p$ their codimension. To achieve this goal, we show how the coefficient of the logarithmic term in the asymptotic expansion of the Schwartz kernel of the Heisenberg operator in the neighborhood of the diagonal relates to the noncommutative residual. Next, we demonstrate that the coefficient of the logarithmic term that we can define in any local coordinate system for the foliation M actually defines a density on M where the density is defined using a symplectic form. Furthemore, we analyze the relation with the Dixmier trace and give the formula of this trace. Finally, we establish a link between this trace of Dixmier and the map res, from the set of compactly based Laplacian-modulated operator to the quotient ℓ^∞/c_0.

2 Heisenberg pseudodifferential operators

Let M be a foliated manifold of dimension n, and let $\mathcal{F}$ be the integrable sub-bundle of the tangent bundle TM of M which defines the foliation. We denote the dimension of the leaves by p, and $q = n - p$ their codimension. Let $(x_1, ..., x_n)$ be a distinguished local coordinate system of M, i.e, the vector fields $\partial x_1, ..., \partial x_p$ (locally) span $\mathcal{F}$, so that $\partial x_{p+1}, ..., \partial x_n$ are transverse to leaves of the foliation. In [6], Connes and Moscovici constructed an algebra of generalized differential operators using Heisenberg calculus. The main idea is that:

. The vector fields $\partial x_1, ..., \partial x_p$ are of order 1

. The vector fields $\partial x_{p+1}, ..., \partial x_n$ are of order 2.

The Heisenberg pseudodifferential calculus consists in defining a class of smooth symbols $\sigma(x, \xi)$ on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ which takes this notion of order into account. For this purpose, one set

$$\| \xi \|^{'} = \left(\xi_1^4 + ... + \xi_p^4 + \xi_{p+1}^2 + ... + \xi_n^2\right)^{\frac{1}{4}}, \text{ for every } \xi \in \mathbb{R}^n,$$
$$\langle\alpha\rangle = \alpha_1 + ... + \alpha_p + 2\alpha_{p+1} + ... + 2\alpha_n, \text{ for every } \alpha \in \mathbb{N}^n.$$

Definition 2.1. *A smooth function $\sigma(x,\xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is a Heisenberg symbol of order $m \in \mathbb{R}$, and we denote $S^m_H(\mathbb{R}^n)$, if σ is x-compactly supported, and if for every multi-index α, β, the following estimate holds :*

$$| \partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi) | \leq \left(1 + \| \xi \|^{'}\right)^{m - \langle\alpha\rangle}.$$

To such a symbol σ of order m, is associated its left-quantization, which is the operator

$$P : C_c^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n),\; Pf(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix.\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi.$$

Here $\hat{f}$ stands of its Fourier transform.
We shall say that P is a Heisenberg pseudodifferential operator of order m, and denote the class of such operators by $\Psi^m_H(\mathbb{R}^n)$. The Heisenberg regularizing operators, whose class is denoted by $\Psi^{-\infty}(\mathbb{R}^n)$, are those of arbitrary order, namely

$$\Psi^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} \Psi^m_H(\mathbb{R}^n).$$

Notice that the Heisenberg regularizing operators are exactly the regularizing operators of the usual pseudodifferential calculus, i.e, the operators with smooth Schwartz kernel.
Actually we shall restrict to the smaller class of classical Heisenberg pseudodifferential operators. For this, we first define the Heisenberg dilations

$$\lambda.(\xi_1, ..., \xi_p, \xi_{p+1}, ..., \xi_n) = (\lambda\xi_1, ..., \lambda\xi_p, \lambda^2\xi_{p+1}, ..., \lambda^2\xi_n)$$

for any non-zero $\lambda \in \mathbb{R}$ and non-zero $\xi \in \mathbb{R}^n$.
Then, a Heisenberg pseudodifferential operator $P \in \Psi^m_H(\mathbb{R}^n)$ of order m is said classical if its symbol σ has an asymptotic expansion

$$\sigma(x,\xi) \sim \sum_{j\geq 0} \sigma_{m-j}(x,\xi) \tag{2.1}$$

where $\sigma_{m-j}(x,\xi) \in \mathcal{S}^{m-j}_H(\mathbb{R}^n)$ are homgeneous, that is, for any non zero $\lambda \in \mathbb{R}$,

$$\sigma_{m-j}(x,\lambda\xi) = \lambda^{m-j}\sigma_{m-j}(x,\xi).$$

The $\sim$ means that for every $M > 0$, there exists an integer N such that $\sigma - \sum_{j=0}^{N} \sigma_{m-j} \in \mathbf{S}^{-M}_H(\mathbb{R}^n)$.
To avoid an overweight of notations, we shall keep the notation Ψ_H to refer to classical elements.
Another important point is the behaviour of symbols towards composition of classical pseudodifferentials operators. Of course, if $P, Q \in \Psi_H(\mathbb{R}^n)$ are Heisenberg pseudodifferential operators of symbols σ_P and σ_Q, PQ is also a Heisenberg pseudodifferential operator of order at most $ord(P) + ord(Q)$, and its symbol σ_{PQ} is given by the following asymptotic expansion called the star-product of symbols, given by the formula

$$\sigma_{PQ}(x,\xi) = \sigma_P \star \sigma_Q(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_P(x,\xi)\partial_x^\alpha\sigma_Q(x,\xi) \tag{2.2}$$

Note that the order of each symbol in the sum is decreasing while $|\alpha|$ is increasing. We define the algebra of Heisenberg formal classical symbols $S_H(\mathbb{R}^n)$ as the quotient

$$S_H(\mathbb{R}^n) \simeq \Psi_H(\mathbb{R}^n)/\Psi^{-\infty}(\mathbb{R}^n).$$

Its elements are formal sums given in (2.1), and the product is the star product (2.2). Note that the $\sim$ can be replaced by equalities when working at a formal level. We now deal with ellipticity in this context. A Heisenberg pseudodifferential operator is said Heisenberg elliptic if it is invertible in the unitalization $S_H(\mathbb{R}^n)^+$ of $S_H(\mathbb{R}^n)$. One can show that this is actually equivalent to say that its Heisenberg principal symbol, e.g the symbol of higher degree in the expansion (2.1) is invertible on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi \setminus \{0\}$. A remarkable specificity of these operators is that they are hypoelliptic, but not elliptic in general. Nevertheless, they remain Fredholm operators between Sobolev spaces relative to this context. The interested reader should consult [2] for details.

Example 2.2. *The following operator, also called sub-elliptic sub-laplacian,*

$$D = \partial^4_{x_1} + ... + \partial^4_{x_p} + \partial^2_{x_{p+1}} + ...\partial^2_{x_n}$$

has Heisenberg principal symbol $\sigma(x,\xi) = \parallel \xi \parallel'^4$, *and is therefore Heisenberg elliptic.*

Remark 2.3. *The analytic dimension of the algebra of Heisenberg differential operators is then* $p + 2q$. *The* p *is the dimension of the leaves,* q *their codimension and* 2 *is the degree of the vector fields transverse to them.*

The following results will be used later. We are questioning whether any foliation is orientable or not.

Definition 2.4. *[12] Let* M *be a foliation of dimension* n*,* p *be the dimension of the leaves and* q *their codimension. This foliation is said to be orientable if there is a volume form on* M*.*

The following proposition gives us an answer.

Proposition 2.5. *[12] The 2-form* ω *is non-degenerate on a foliation of dimension* $2n$ *if and only if* $\omega^n = \omega \wedge ... \wedge \omega$ *is non-zero at all points.*

Any symplectic foliation (M,ω) is canonically oriented by its symplectic structure. The form $\frac{\omega^n}{n!}$ is called the symplectic volume or the Liouville volume of (M,ω).
Consider the $(n-1)-$ form on $\mathbb{R}^n$, $n \geqslant 2$, given by, (see [17])

$$\sigma(\xi) = \iota_X \Omega(\xi) = \sum_{j=1}^{n} (-1)^{j+1} \xi_j d\xi_1 \wedge ... \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge ... \wedge d\xi_n. \tag{2.3}$$

Here, X $= \sum_{i=1}^{p} \xi_i \partial_{\xi_i} + 2 \sum_{i=p+1}^{n} \xi_i \partial_{\xi_i}$ is the generator of the Heisenberg dilations, ι stands for the interior product and $\Omega(\xi) := d\xi_1 \wedge ... \wedge d\xi_n$ denotes the volume form on $\mathbb{R}^n$.
Denoting by ω the canonical symplectic form on T^*M and by ρ the radial vector field one has

$$a_{-(p+2q)} \iota_X \Omega(\xi) \wedge dx_1 \wedge ... \wedge dx_n = (-1)^{n\frac{n-1}{2}} \frac{1}{n!} (a_{-(p+2q)} \iota_X \omega^n), \tag{2.4}$$

where $a_{-(p+2q)}$ is the Heisenberg homogeneous term of order $-(p+2q)$ in the asymptotic expansion of a into homogeneous forms.

In what follows, let us recall the notion of holomorphic families of Heisenberg pseudifferential operators. Let $\mathcal{D}$ be an open domain in $\mathbb{C}$ and U an open subset on $\mathbb{R}^{p+q}$ equiped with an integrable sub-bundle $\mathcal{F} \subset TU$ and $\mathcal{F}-$frame $X_1, X_2, ..., X_n$ of TU.

Definition 2.6. *[19] Let $z \to m(z)$ be a holomorphic function from $\mathcal{D}$ to $\mathbb{C}$. A family $z \to a(x, \xi, z) \in C_c^\infty(U \times \mathbb{R}^{p+q} \times \mathcal{D})$ is said to be a holomorphic family of Heisenberg classical symbols of order m on $\mathcal{D}$, and we denote $a(x, \xi, z) \in \mathbb{S}^m_{hom}(U, \mathcal{D})$ if:*

1. *the map $z \to a(x, \xi, z)$ is holomorphic for all fixed (x, ξ);*
2. *there exist for all j some functions $z \to a_{m(z)-j}(x, \xi, z)$ positively homogeneous in ξ of degree $m(z) - j$ such that:*

(a) *the map $z \to a_{m(z)-j}(x, \xi, z)$ is holomorphic from $\mathcal{D}$ into the set $C^\infty(S^*U)$ equipped with the topology of uniform convergence of functions and all their derivatives on any compact set;*

(b) *for all $N \in \mathbb{N}$, for all multi-indices α, β, for all compacts $K \subset U$ and $K' \subset \mathcal{D}$, there exists a constant $C = C(N, \alpha, \beta, K, K')$ such that:*

$$\left| \partial_\xi^\alpha \partial_x^\beta \left[a(x, \xi, z) - \chi(\xi) \sum_{j=0}^{N-1} a_{m(z)-j}(x, \xi, z) \right] \right| \leq C \parallel \xi \parallel'^{m-N-\langle\alpha\rangle}$$

for $(x, \xi, z) \in K \times \mathbb{R}^{p+q} \times K'$ and $m \geq |m(z)|$ on K'. Here $\chi(\xi) = 0$ for $\parallel \xi \parallel' \leq \frac{1}{2}$ and $\chi(\xi) = 1$ for $\parallel \xi \parallel' \geq 1$ and S^*U is the cosphere bundle of U.

Now that we have defined the holomorphic families of Heisenberg symbols, we can hold the holomorphic families of Heisenberg pseudodifferential.

Definition 2.7. *[19] Let $m : z \to m(z)$ be holomorphic function from $\mathcal{D}$ to $\mathbb{C}$.*

1. *A family $z \to A(z)$ is called a holomorphic family of order $m(z)$ on $\mathcal{D}$, and we denote $A \in \mathbb{P}^m(U, \mathcal{D})$, if the family $A(z) \in \Psi_H^{m(z)}(U)$ is a family of Heisenberg pseudodifferential operators with compact support in the open set U such that the symbol of the operator $A(z)$ is given by an element $a(x, \xi, z) \in \mathbb{S}^m_{hom}(U, \mathcal{D})$.*
2. *Let M be compact manifold. A family $A(z) \in \mathcal{P}(M)$ is called holomorphic of order m if in any local coordinate system U, any function $\phi \in C_c^\infty(U)$, we can write $A(z)\phi = A_\phi(z) + R_\phi(z)$, with $A_\phi \in \mathbb{P}^m(U, \mathcal{D})$ and $z \to R_\phi(z)$ is a holomorphic family of Heisenberg regularizing operators, whose kernel K_ϕ belongs to $C^\infty(M \times M \times \mathcal{D})$. We denote $\mathbb{P}^m(M, \mathcal{D})$ the class of these operators.*

3 Construction of the noncommutative residue for the symplectic foliation

In this section we will revisite the construction of the noncommutative residue. For this purpose we show, following the approach of [16] and [17], that the kernel of a Heisenberg pseudodifferential

operator admits near the diagonal an asymptotic expansion whose coefficient of the logarithmic term is an intrinsic density. We shall start by extending an a priori homogeneous symbol defined on $\mathbb{R}^{p+q} \setminus \{0\}$ into a homogeneous distribution on $\mathbb{R}^{p+q}$.
Let us recall that $\mathbb{R}_+^*$ acts on the Schwartz space $S(\mathbb{R}^{p+q})$ by dilation in the following way :

$$(\lambda.f)(x) = f_\lambda(x) = f(\lambda x).$$

This action extends by duality to $S'(\mathbb{R}^{p+q})$ and $\lambda.\tau = \tau_\lambda$ is denoted by the equality

$$\langle \tau_\lambda, f\rangle = \lambda^{-(p+2q)} \langle \tau, f_{\lambda^{-1}}\rangle \text{ for } \lambda > 0 \text{ and } f \in S(\mathbb{R}^{p+q}).$$

Definition 3.1. *We call homogeneous distribution of degree $m \in \mathbb{C}$ on $\mathbb{R}^p$ a distribution τ such that $\tau_\lambda = \lambda^m \tau$ for $\lambda > 0$*

Lemma 3.2. *Let $\sigma \in C^\infty(\mathbb{R}^{p+q} \setminus 0)$ be a homogeneous symbol of order $m \in \mathbb{C}$.*

1. *if $m \in \mathbb{Z}$ and $m > -(p+2q)$, then σ extends to a homogeneous tempered distribution on $\mathbb{R}^{p+q} \setminus 0$.*

2. *if $m \in \mathbb{C} \setminus \mathbb{Z}$, then σ has an unique homogeneous extension as a tempered distribution .*

3. *if $m = -(p+2q)$, there is a unique obtruction to extend σ into a homogeneous distribution,*

$$c_\sigma = \int_{\|\xi\|'=1} \sigma(\xi)\iota_X(\Omega)(\xi) \tag{3.1}$$

more precisely one can extend σ into a tempered distribution τ satisfying

$$\tau_\lambda = \lambda^{-(p+2q)}\tau + c_\sigma \lambda^{-(p+2q)} \log \lambda \delta_0 \quad \forall \lambda > 0. \tag{3.2}$$

Proof.

1. The function σ is integrable near the origin because $m > -(p+2q)$ and C^∞ with polynomial growth at infinity. It therefore naturally to define a tempered distribution, which is homogeneous.

2. Let us fix a positive integer l such that $l > \Re m + p + 2q + 1$. Following the method developped by R. Ponge in [16], let us define a function $\psi \in C_c^\infty(\mathbb{R}_+)$ worth 1 in the neighborhood of 0. We then extend σ to a tempered distribution τ by setting:

$$\langle \tau, f\rangle = \int_{\mathbb{R}^{p+q}} \left[f(\xi) - \psi(\| \xi \|') \sum_{\langle\alpha\rangle \leqslant l} \frac{f^{(\alpha)}(0)\xi^\alpha}{\alpha!} \right] \sigma(\xi) d\xi, \quad f \in S(\mathbb{R}^{p+q}) \tag{3.3}$$

Let λ be a postive real number . Then

$$\begin{aligned}\langle\tau_\lambda, f\rangle - \lambda^m\langle\tau, f\rangle &= \lambda^{-(p+2q)}\int_{\mathbb{R}^{p+q}}\left[f(\lambda^{-1}\xi) - \psi(\|\xi\|')\sum_{\langle\alpha\rangle\leqslant l}\frac{f^{(\alpha)}(0)(\lambda^{-1}\xi)^\alpha}{\alpha!}\right]\sigma(\xi)d\xi\\ &\quad -\lambda^m\int_{\mathbb{R}^{p+q}}\left[f(\xi)-\psi(\|\xi\|')\sum_{\langle\alpha\rangle\leqslant l}\frac{f^{(\alpha)}(0)\xi^\alpha}{\alpha!}\right]\sigma(\xi)d\xi\\ &= \lambda^m\sum_{\langle\alpha\rangle\leqslant l}\frac{f^{(\alpha)}(0)}{\alpha!}\int_{\mathbb{R}^{p+q}}\left[\psi(\|\xi\|')-\psi(\lambda\|\xi\|')\right]\xi^\alpha\sigma(\xi)d\xi\end{aligned}$$

Changing to polar coordinates and using the homogeneity of σ, we have

$$\begin{aligned}&\int_{\mathbb{R}^{p+q}}\left[\psi(\|\xi\|')-\psi(\lambda\|\xi\|')\right]\xi^\alpha\sigma(\xi)d\xi\\ &=\int_0^\infty\int_{\|\xi\|'=1}\mu^{(p+2q)-1}\left(\psi(\mu)-\psi(\lambda\mu)\right)(\mu\xi)^\alpha\sigma(\mu\xi)d\mu d\xi\\ &=\int_0^\infty\int_{\|\xi\|'=1}\mu^{m+(p+2q)+\langle\alpha\rangle-1}\left(\psi(\mu)-\psi(\lambda\mu)\right)\xi^\alpha\sigma(\xi)d\mu d\xi\\ &=\int_0^\infty\mu^{m+(p+2q)+\langle\alpha\rangle-1}\left(\psi(\mu)-\psi(\lambda\mu)\right)d\mu\int_{\|\xi\|'=1}\xi^\alpha\sigma(\xi)d\xi\\ &=\int_0^\infty\mu^{m+(p+2q)+\langle\alpha\rangle}\left(\psi(\mu)-\psi(\lambda\mu)\right)\frac{d\mu}{\mu}\int_{\|\xi\|'=1}\xi^\alpha\sigma(\xi)\iota_X\Omega(\xi)\end{aligned}$$

hence

$$\langle\tau_\lambda, f\rangle - \lambda^m\langle\tau, f\rangle = \lambda^m\sum_{\langle\alpha\rangle\leqslant l}\frac{f^{(\alpha)}(0)}{\alpha!}c_{\xi^\alpha\sigma}\rho_\alpha(\lambda),$$

with

$$c_{\xi^\alpha\sigma} = \int_{\|\xi\|'=1}\xi^\alpha\sigma(\xi)\iota_X(\Omega)(\xi)\text{ and }\rho_\alpha(\lambda)=\int_0^\infty\mu^{m+(p+2q)+\langle\alpha\rangle}\left(\psi(\mu)-\psi(\lambda\mu)\right)\frac{d\mu}{\mu}.$$

We have $\rho_\alpha(1)=0$ and $\frac{d}{d\lambda}\rho_\alpha(\lambda) = -\lambda^{m+(p+2q)+\langle\alpha\rangle-1}\displaystyle\int_0^\infty\mu^{m+(p+2q)+\langle\alpha\rangle}\psi'(\mu)d\mu$. Thus τ is homogeneous if, and only if, we have

$$\int_0^\infty t^a\psi'(t)=0 \qquad for \quad a=m+p+2q,...,m+p+2q+l. \tag{3.4}$$

We are therefore reduced to looking for a function ψ that satisfies the condition above. Let's set ψ in the form

$$\psi(\mu) = h(\log \mu),$$

with $h \in C^\infty(\mathbb{R})$ equal 1 near $-\infty$ and 0 near $+\infty$. In that case the condition (3.4) becomes

$$0 = \int_0^\infty \mu^a \frac{d}{d\mu}(h(\log \mu))d\mu = \int_{-\infty}^\infty e^{as}h'(s)ds$$

for $a = m + (p + 2q), ..., m + (p + 2q) + l$. Now, let $g \in C_c^\infty(\mathbb{R})$ such that

$$\int e^{at}(\frac{d}{dt} + a)g(t)dt = \int \frac{d}{dt}(e^{at}g(t))dt = 0,$$

$$\int (\frac{d}{dt} + a)g(t)dt = a \int g(t))dt.$$

Since $m + (p + 2q), ..., m + (p + 2q) + l$ are non-zero, we see that if $\int g(t)dt = 1$, the condition (3.4) is verified by $\psi(\mu) = h(\log \mu)$ if $h'(s)$ is given by

$$h'(s) = \left(\prod_{a=m+p+2q}^{m+p+2q+l} (a^{-1}\frac{d}{dt} + 1) \right) g(s), \qquad s \in \mathbb{R}. \tag{3.5}$$

In that case the distribution τ given by (3.3) is a homogeneous extension of σ. At last, if τ' is another homogeneous distribution extending σ then, $\tau - \tau'$ is supported in $\{0\}$ and homogeneous of degree not integer. Now this is only possible if $\tau = \tau'$, so τ is the unique homogeneous extension of σ.

3. Now, assume that $m = -(p + 2q)$. As a distribution on $\mathbb{R}^{p+q} \setminus \{0\}$ the symbol σ extends into a continuous linear form

$$L(f) = \int_{\mathbb{R}^{p+q}} f(\xi)\sigma(\xi)d\xi,$$

which is defined on the closed subspace :

$$S_0 = \{f \in S(\mathbb{R}^{p+q}); f(0) = 0\}.$$

Using the Hanh-Banach theorem we can extend L into a continuous linear form on S and thus into a tempered distribution. Such a distribution belongs to the affine subspace

$$E = \left\{\tau \in S'; \tau_{|S_0(\mathbb{R}^{p+q})} = L\right\},$$

whose direction is the line generated by Dirac mass at zero (δ_0). Let $\lambda > 0$ and $\tau \in E$. if $f \in S_0$, then $f_{\lambda^{-1}} \in S_0$ and we have:

$$\lambda^{p+2q} \langle \tau_\lambda, f \rangle = \langle \tau, f_{\lambda^{-1}} \rangle = L(f_{\lambda^{-1}}).$$

The homogeneity of σ implies:

$$L(f_{\lambda^{-1}}) = \int_{\mathbb{R}^{p+q}} f(\lambda^{-1}\xi)\sigma(\xi)d\xi = \int_{\mathbb{R}^{p+q}} f(\xi)\sigma(\xi)d\xi = L(f)$$

Thus E is stable by endomorphism $\tau \to \lambda^{p+2q}\tau_\lambda$. Since $\lambda^{p+2q}(\delta_0)_\lambda = \delta_0$, we deduce that there exists $c(\lambda) \in \mathbb{C}$ such that

$$\tau_\lambda = \lambda^{-(p+2q)}\tau + c(\lambda)\lambda^{-(p+2q)}\delta_0, \text{ for all } \tau \in E.$$

To determine $c(\lambda)$, we just need to compute $\tau_\lambda - \lambda^{-(p+2q)}\tau$ for a particular element of E. We obtain a distribution $\tau \in E$ for example by considering $\psi \in C_c^\infty([0,\infty[)$ such that 1 near 0 and defining τ by :

$$\tau(f) = \int (f(\xi) - f(0)\psi(\| \xi \|'))\sigma(\xi)d\xi, \quad f \in S. \tag{3.6}$$

Let $f \in S$ such that $f(0) = 1$. Then :

$$\begin{aligned}
c(\lambda) &= \tau(f_\lambda) - \tau(f) \\
&= \int_{\mathbb{R}^{p+q}} (f(\lambda^{-1}\xi) - \psi(\| \xi \|'))\sigma(\xi)d\xi - \int_{\mathbb{R}^{p+q}} (f(\xi) - \psi(\| \xi \|'))\sigma(\xi)d\xi \\
&= \int (\psi(\lambda \| \xi \|') - \psi(\| \xi \|'))\sigma(\xi)d\xi \\
&= c_\sigma \int_0^\infty (\psi(\lambda\mu) - \psi(\mu))\frac{d\mu}{\mu}, \quad c_\sigma = \int_{\|\xi\|'=1} \sigma(\xi)\iota_X(\Omega)(\xi).
\end{aligned}$$

Since

$$\lambda\tfrac{d}{d\lambda}\int_0^\infty (\psi(\lambda\mu) - \psi(\mu))\frac{d\mu}{\mu} = \lambda\int_0^\infty \psi'(\lambda\mu)d\mu = \psi(0) = 1$$

we see that $c(\lambda) = c_\sigma \log\lambda$. Thus:

$$\tau_\lambda = \lambda^{-(p+2q)}\tau + c_\sigma\lambda^{-(p+2q)}\log\lambda\delta_0 \text{ for all } \tau \in E$$

It follows that if $c_\sigma = 0$, any element of E is a homogeneous distribution on $\mathbb{R}^{p+q}$ extensing the symbol σ. Reciprocally, let $\tau \in S'$ be a homogeneous distribution extending σ and let $\tau' \in E$. Since the support of $\tau - \tau'$ is included in $\{0\}$, we have $\tau = \tau' + \sum_{|\alpha|\le N} a_\alpha\partial^\alpha\delta_0$ avec $a_\alpha \in \mathbb{C}$. Thus

$$0 = \tau_\lambda - \lambda^{-(p+2q)}\tau = c_\sigma\lambda^{-(p+2q)}\log\lambda\delta_0 + \sum_{1\le|\alpha|\le N} a_\alpha\lambda^{-(p+2q)}(\lambda^{|\alpha|} - 1)\partial^\alpha\delta_0,$$

which implies that $c_\sigma = 0$ and $\tau \in E$. Thus the condition $c_\sigma = 0$ is necessary and sufficient condition to extend into a homogeneous distribution. In the general case we can at best be extended it into a distribution satisfying (3.2), which is only fulfilled by the element of E.

□

Proposition 3.3. *[6] Let P be a Heisenberg pseudodifferential operator of order $-(p+2q)$. Then the kernel $K(x,y)$ for P has the following behavior near the diagonal*

$$K(x,y) = c(x)\log(\| x - y \|') + 0(1),$$

where the $1-$*density* $c(x)$ *is given by the formula*

$$c(x) = (2\pi)^{-(p+q)} \int_{\|\xi\|'=1} \sigma(x,\xi)\iota_X \Omega(\xi). \tag{3.7}$$

We define the noncommutative residue for foliated manifold as follows:

Definition 3.4. *Let* (M,ω) *be a foliated manifold. The noncommutative residue of a Heisenberg pseudodifferential operator* $P \in \Psi_H^{\mathbb{Z}}(M)$ *is the linear functional defined by*

$$resP = (2\pi)^{-(p+q)} \int_M \int_{\|\xi\|'=1} \sigma_{-(p+2q)} \iota_X \Omega(\xi) dx. \qquad P \in \Psi_H^{\mathbb{Z}}(M) \tag{3.8}$$

Proposition 3.5. *For any* $\phi \in C_c^\infty(\mathbb{R}^n)$ *one has*

$$res(\phi(X)(1-D)^{-\frac{n}{4}}) = (2\pi)^{-n} \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \int_{\mathbb{R}^n} \phi(x)dx,$$

where D *is the sublaplacian defined in example (2.2), regarded as a compact operator on* $L^2(\mathbb{R}^n)$ *and* $X = (X_1, \cdots, X_d)$ *with* X_j *the self-adjoint operator of multiplication by the variable* x_j*.*

Proof. For ϕ in $C_c^\infty(\mathbb{R}^n)$ let us consider the operator $\phi(X)(1-D)^{\frac{-n}{4}}$ which is associated with a classical and compactly based symbol of order $-n$. Its principal symbol is given for $|\xi| > 1$ by the map $(x,\xi) \to \phi(x)(2\pi \parallel \xi \parallel')^{-n}$. Thus, we easily get

$$\begin{aligned} res(\phi(X)(1-D)^{-n/4}) &= \int_{\mathbb{R}^n} \int_{\|\xi\|'=1} \phi(x)(2\pi)^{-n} \iota_X \Omega(\xi) dx \\ &= (2\pi)^{-n} \int_{\|\xi\|'=1} \iota_X \Omega(\xi) \int_{\mathbb{R}^n} \phi(x)dx \\ &= (2\pi)^{-n} vol(S^n) \int_{\mathbb{R}^n} \phi(x)dx, \end{aligned}$$

where $vol(S^n)$ denotes the volume of the sphere S^n.

Let's show that $vol(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$.
Let's put $\int_{\mathbb{R}^{n+1}} e^{-|x|^2} dx = (\pi)^{\frac{n+1}{2}}$.

Using the formula for integration in polar coordinates

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} e^{-|x|^2} dx &= \int_0^\infty \int_{\|\xi\|'=1} e^{-r^2} r^n dr d\sigma \\ &= \int_{\|\xi\|'=1} d\sigma \int_0^\infty e^{-r^2} r^n dr \\ &= \int_{\|\xi\|'=1} \iota_X \Omega(\xi) \int_0^\infty e^{-r^2} r^n dr \\ &= vol(S^n) \int_0^\infty e^{-r^2} r^n dr. \end{aligned}$$

One has

$$\int_0^{\infty} e^{-r^2} r^n dr = \frac{1}{2}\int_0^{\infty} e^{-t} t^{\frac{d-1}{2}} dt$$
$$= \frac{1}{2}\Gamma(\frac{n+1}{2}).$$

Finally, $vol(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$.

□

Definition 3.6. *We call symplectomorphism a diffeomorphism ϕ between two symplectic manifolds (M, ω) and (M', ω') such that $\phi^*\omega' = \omega$.*

Proposition 3.7. *Let M be a compact foliation of dimension n. Then :*
The density defined in (3.7) is invariant by symplectomorphism, i.e that if $\phi : (M, \omega) \to (M', \omega')$ is a symplectomorpism we have

$$c_{\phi^* P}(x) = |J_\phi(x)| c_P(\phi(x))$$

with J_ϕ the associated jacobien at ϕ and P is a Heisenberg pseudodifferential operator.

Proof. Given a local chart change $x = \phi^{-1}(x)$. The kernel of $K_{\phi^* P}(x, y)$ is that of $\phi^* P$. It is therefore equal, by change of variable to $|J_\phi(y)| K_P(\phi(x), \phi(y))$.
So we have

$$K_{\phi^* P}(x, y) = |J_\phi(y)| c_P(\phi(x)) \log \parallel \phi(x) - \phi(y) \parallel' + 0(1).$$

Furthermore

$$|J_\phi(y)| c_P(\phi(x)) \log \parallel \phi(x) - \phi(y) \parallel' = |J_\phi(x)| c_P(\phi(x)) log \parallel x - y \parallel' + 0(1).$$

We finally get $c_{\phi^* P}(x) = |J_\phi(x)| c_P(\phi(x))$. This shows that $res_x(P) = c_P(x)dx$ defines a density on M. □

Lemma 3.8. *[6] The functional $\sigma \to L(\sigma) = (2\pi)^{-(p+q)} \int \sigma(\xi)$ has a unique holomorphic extension $\tilde{L}$ to the space of symbols of non integral order $z \notin \mathbb{Z}$. The value of $\tilde{L}$ on $\sigma \sim \sum \sigma_{z-p}$, is given by*

$$\tilde{L}(\sigma) = (2\pi)^{-(q+p)} \int \left(\sigma(\xi) - \sum_{j=0}^{N} \tau_{z-j}(\xi) \right) d\xi, \; N \geq \Re(z) + (p + 2q)$$

where τ_{z-p} is the unique homogeneous extension of σ_{z-j} and σ is a classical symbol of order z.

Complex Powers of elliptic sublaplacians. The results of Wodzicki can be generalized to the context of foliations, see [14], [18] and [6] . Thanks to a partition of unity, we can construct a sub-elliptic sub-Laplacian D from Example 2.2. Its complex powers D^{-s} are defined as the following eigensupport pseudodifferential operators, using the parametrix $(\lambda - D)^{-1}$ and an appropriate Cauchy integral

$$D^{-s} = \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} (D - \lambda)^{-1} d\lambda, \quad \Re s > 0,$$

$D^{-s} = D^k D^{-s-k}$, $\Re(s) + k > 0$,$k \in \mathbb{N}$. Here $\mathcal{C}$ is the path in $\mathbb{C}$ going from infinity along $R_\theta = \{s \in \mathbb{C} : s = re^{i\theta}, r \geq 0\}$ to a small circle around 0, clockwise about the circle, and back along R_θ.

Theorem 3.9. *[16] The family $(D^{-s})_{\Re s>0}$ of the complex powers of D is a holomorphic family of the Heisenberg pseudodifferential operators.*

Proposition 3.10. *[6] The Zeta function $s \to Tr(AD^{-s})$ is holomorphic for $4\Re s > (m+p+2q)$ and extends uniquely to a holomorphic function $s \to TR(AD^{-s})$ for $s \in \mathbb{C} \setminus \mathbb{Z}$, where A is a Heisenberg pseudodifferential operator of order m and D a sub-elliptic sub-laplacian defined in example 2.2.*

Proof. In a local chart, the trace of a Heisenberg pseudodifferential operator of order $< -(q + 2p)$ is given by

$$Tr(A) = (2\pi)^{-(p+q)} \int \sigma(x, \xi) dx d\xi \quad , \tag{3.9}$$

where the total symbol σ is smooth. Thus, in a local chart, the following formula provides the required extension of Trace to Heisenberg pseudodifferential operator of order $s \notin \mathbb{Z}$

$$TR(A) = \int \tilde{L}(\sigma(x, .))dx. \tag{3.10}$$

□

Theorem 3.11. *Let $(M, \mathcal{F})$ be a foliated manifold of dimension n, where $\mathcal{F} \subset TM$ is the integrable defining the foliation, p be the dimension of the leaves, q their codimension and $A \in \Psi^m_{H,C}(M)$ be a Heisenberg pseudodifferential operator with compact support of order $m \in \mathbb{R}$. Let D be the sub-elliptic sub-laplacian defined in example 2.2. Then, the zeta function $\zeta(s) = Tr(AD^{-s})$ is holomorphic for $4\Re s > m + p + 2q$, and extends to meromorphic function on the complex plane, with at most simple poles in the set $\{\frac{m+p+2q}{4}, \frac{m+p+2q-1}{4}, ...\}$ the residue reads*

$$resA = 4Res_{s=0}\ Tr(AD^{-s}) = \int_M c(x),$$

where $c(x)$ is defined in (3.7)

Proof. We can work at the level of symbols and consider a fixed symbol σ of integral order m. We let $\sigma_s(\xi) = \sigma(\xi)(\| \xi \|')^{-4s}$ and investigate the behavior of $\tilde{L}(\sigma_s)$ near $s = 0$. Let $4N \geq m+(p+2q)$, then

$$\tilde{L}(\sigma_s) = (2\pi)^{-(q+p)} \int \left(\sigma(\xi) - \sum_0^N \sigma_{m-k}(\xi) \right) (\| \xi \|')^{-4s} d\xi$$

where $\sigma_{m-k}(\xi)(\| \xi \|')^{-4s}$ is replaced near 0 by its unique extension as a homogeneous distribution. The singularity at $s = 0$ comes from ξ in the neighborhood of 0. When $m-k > -(p+2q), \sigma_{m-k}(\xi)d\xi$ is integrable at 0 and the unique extension of $\sigma_{m-k}(\| \xi \|')^{-4s}d\xi$ is holomorphic in s at $s = 0$. Thus none of thes terms contribute to the singularity of $\tilde{L}(\sigma_s)$ at $s = 0$. We can choose $4N = m+(p+2q)$ since, by Lemma 3.8, any larger value gives the same answer. We thus need to understand the behavior at $s = 0$ of

$$\int_{\|\xi\|' \leq 1} \sigma_{-(p+2q)}(\xi) \| \xi \|'^{-4s} d\xi \tag{3.11}$$

where $\sigma_{-(p+2q)}(\xi)(\| \xi \|')^{-4s}d\xi$ is extended uniquely as a homogeneous distribution at $\xi = 0$. But for $Res < 0$ one has integrability near 0 so that this unique extension is the obvious one and one can write (3.11) as

$$\int_{\|\xi\|'=1} \sigma_{-(p+2q)}\iota_X(\Omega)(\xi) \int_0^1 u^{-4s}\frac{du}{u}. \tag{3.12}$$

As $\int_0^1 u^{-4s}\frac{du}{u} = \left[\frac{u^{-4s}}{s}\right]_0^1 = -\frac{1}{s}$, one gets that the singularity of $\tilde{L}(\sigma_s)$ at $s = 0$ is a simple pole with residue:

$$\frac{(2\pi)^{-(p+q)}}{4}\int_{\|\xi\|'=1} \sigma_{-(p+2q)}\iota_X(\Omega)(\xi). \tag{3.13}$$

□

Theorem 3.12. *Let M be a foliated manifold of dimension n, p be the dimension of the leaves, q their codimension and $A \in \Psi_H(M)$. Assume M is equipped with a symplectic form ω. We have the following formula, only depending on the symbol σ of A.*

$$resA = (2\pi)^{-n}\int_M \int_{\|\xi\|'=1} \iota_X\left(\sigma_{-(p+2q)}(x,\xi)\frac{\omega^n}{n!}\right).$$

Proof. Let A be a Heisenberg pseudodifferental operator of order $m \in \mathbb{Z}$ with symbol $\sigma(x,\xi) \sim \sum_{j=0}^{\infty}\sigma_{m-j}(x,\xi)$.

We define the local density $res_xA, \quad x \in M$, by

$$res_xA = \left(\int_{\|\xi\|'=1} \sigma_{-(p+2q)}\iota_X\Omega(\xi)\right) dx_1 \wedge dx_2 \wedge ... \wedge dx_n. \tag{3.14}$$

The definition of the noncommutative residue is given as follows

$$resA = \int_M res_xA.$$

By the formula 2.3 one has

$$resA = (2\pi)^{-n}\int_M \int_{\|\xi\|'=1} \iota_X\left(\sigma_{-(p+2q)}(x,\xi)\frac{\omega^n}{n!}\right).$$

□

We are now going to show that the noncommutative residue makes it possible to define a trace on the algebra of Heisenberg pseudodifferential operators of integer order.

Lemma 3.13. *[19] Let p_{-n} be a derivative*

$$p_{-n} = \frac{\partial}{\partial\xi_k}p_{-n+1},$$

where p_{-n+1} is a smooth homogeneous function on $\mathbb{R}^n \setminus \{0\}$ of degree $-n+1$. Then

$$\int_{\|\xi\|'=1} p_{-n}\iota_X\Omega(\xi) = 0.$$

Lemma 3.14. *[19] Let p be a hommogeneous function on $\mathbb{R}^n\setminus\{0\}$. Each of the following conditions is sufficient for p to be a sum of derivatives:*

1. *deg* $p \neq -n$
2. *deg* $p = -n$ *and* $\int_{\|\xi\|'=1} p_{-n}\iota_X\Omega(\xi) = 0$
3. $p = \xi^\alpha \partial^\beta q$ *where q is a homogeneous function and* $|\beta| > |\alpha|$.

Proposition 3.15. *Let M be a foliated compact manifold of dimension n. Then:*

1. *The noncommutative residue is invariant by symplectomorphism, i.e, if $\phi : M \to M'$ is a symplectomorphism one has:*

$$res_M P = res_{M'}\phi^* P \qquad \forall P \in \Psi_H^{\mathbb{Z}}(M).$$

2. *The noncommutative residue defines a trace on the algebra* $\Psi_H^{\mathbb{Z}}(M)$.

3. *Any trace defined on the algebra $\Psi_H^{\mathbb{Z}}(M)$ coincides with the trace res up to multiplication by a constant, if M is closed of dimension $n > 1$.*

Proof.

1. Under a change of variables $x = \phi(y)$ the symbol $p(x,\xi)$ transforms to a symbol $\bar{p}$ according to the formula

$$\bar{p}(y,{}^t\phi^*(y)\xi) \sim \sum_{|\alpha|\geq 0} \partial_\xi^\alpha p(\phi(y),\xi)\varphi_\alpha(y,\xi), \tag{3.15}$$

where $\phi_\alpha(y,\xi)$ are polynomials in ξ of degree $\leq \frac{|\alpha|}{2}$ and $\varphi_0 = 1$ (see Hormander [13], formula (18.1.30)). Using a linear change of variables and 3.15 we get

$$\int_{\|\xi\|'=1} \bar{p}(y,\eta)\sigma_\eta = |det\phi^*(y)| \int_{\|\xi\|'=1} p_{-n}(\phi(y),\xi)\sigma_\xi, \qquad where \quad \eta = \phi\xi \tag{3.16}$$

since the terms with $\mid \alpha \mid > 0$ do not contribute to the integral in virtue of Lemma 3.14(3). The transformation law (3.16) shows that expression (3.14) is indeed a density on M, so that $resA$ is well-defined. We may proceed considering the operators whose symbols have supports in a fixed coordinate chart. The general case may be reduced to this special one using a partition of unity since the density res_x P does not depend on the choice of local coordinates.

2. To prove (2), consider two operators P, Q with symbols p and q supported in a coordinate chart U. Without loss of generality, we shall assume that U is diffeomorphic to an open ball of $\mathbb{R}^n$. The symbol of $[P,Q]$ is given by

$$\sum_{|\alpha|\geq} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha p \partial_x^\alpha q - \partial_\xi^\alpha q \partial_x^\alpha p). \tag{3.17}$$

This expression may be represented as a sum of derivatives

$$\sum_{j=0}^{n} \frac{\partial}{\partial \xi_j} A_j + \frac{\partial}{\partial x_j} B_j, \tag{3.18}$$

where A_j and B_j are bilinear expressions in p and q and their derivatives. In particular, they have compact supports contained in U. Thus, the integrals over $\parallel \xi \parallel' = 1$ of $((\partial/\partial\xi_j)A_j)_{-n}$ vanish by Lemma 3.13, while the integrals of $(\partial/\partial\xi_j)A_j)_{-n}$ over U vanish, since all B_j have compact support in U. This proves (2).

We will need the explicit expressions of A_j and B_j for the terms in (3.17) with $|\alpha| = 1$, that is

$$-i\sum_{k=1}^{n} \frac{\partial p}{\partial \xi_k}\frac{\partial q}{\partial x_k} - \frac{\partial q}{\partial \xi_k}\frac{\partial p}{\partial x_k} = -i\sum_{k=1}^{n} \frac{\partial}{\partial \xi_k}\left(p\frac{\partial q}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(p\frac{\partial q}{\partial \xi_k}\right) \tag{3.19}$$

Finally, to prove uniqueness, consider an operator P with symbol p supported in a coordinate chart U and let $\bar{x}_j$ and $\bar{\xi}_j$ denote any symbols with supports in U coinciding with x_j and ξ_j on the support of p. Then, taking $q = \bar{x}_j$ or $q = \bar{\xi}_j$ in (3.19), we obtain

$$[p, \bar{x}_j] = -i\frac{\partial p}{\partial \xi_j}; \quad [p, \bar{\xi}_j] = i\frac{\partial p}{\partial x_j} \tag{3.20}$$

3. Given a trace τ on the whole algebra of complete symbols the equalities (3.20) imply that

$$\tau\left(\frac{\partial p}{\partial \xi_j}\right) = \tau\left(\frac{\partial p}{\partial x_j}\right) = 0 \tag{3.21}$$

since the trace must vanish on commutators. Let $p \sim \sum_{k\leq m} p_k \in \Psi_H^{\mathbb{Z}}$ and define $\bar{p}_{-n}(x) = (1/\iota_X\Omega \int_{\|\xi\|'=1} p_{-n}(x,\xi)\sigma_\xi$. Applying Lemma 3.14(1) to p_k for all $k \neq -n$, there exist n functions $q_k^{(j)}(x,\xi)$, $1 \leq j \leq n$, homogeneous of degree $k+1$ in ξ such that $p_k = \sum_{j=1}^{n} \partial\xi_j q_k^{(j)}$. Define, for all, $1 \leq j \leq n$, $b_j(x,\xi) \sim \sum_{k\leq, k\neq -n} q_k^{(j)}$. One has

$$p(x,\xi) - \bar{p}_{-n}(x) \mid \xi \mid^{-n} = \sum_{j=1}^{n} \partial_{\xi_j} b_j(x,\xi) + p_{-n}(x,\xi) - \bar{p}_{-n}(x) \mid \xi \mid^{-n}.$$

Since

$$\int_{\|\xi\|'=1} \left(p_{-n}(x,\xi) - \bar{p}_{-n}(x) \mid \xi \mid^{-n}\right)\sigma_\xi = 0,$$

Lemma 3.14 (2) shows that the expression $p_{-n}(x,\xi) - \bar{p}_{-n}(x) \mid \xi \mid^{-n}$ is a (finite) sum of derivatives in the variable ξ. Putting this together, it follows that

$$\tau(p) = \tau\bar{p}_{-n}(x) \mid \xi \mid^{-n}.$$

Now, the map $C_0^\infty(U) \ni f \to \mu(f) = \tau(f \mid \xi \mid^{-n}$ defines a C-linear form on $C_0^\infty(U)$; it follows from (3.20) above that $\mu(\partial_{x_j} f) = 0$ for all $1 \leq j \leq n$ and $f \in C_0^\infty(U)$. Hence, since U is symplectomorphism to an open ball of $\mathbb{R}^n$, there exists $c \in \mathbb{C}$ such that $\mu(f) = c \int_U f(x)dx$ for all $f \in C_0^\infty(U)$.

□

4 Modulated operators

In this section, We begin by reviewing the definition of Dixmier trace; for more details see [3]. Then we introduce the concept of modulated operators and we establish a link between these modulated operators and the Dixmier trace.

Let H be an (infinite-dimensional) Hilbert space, $T \in \mathbf{K}(H)$; and $|T| = (T^*T)^{\frac{1}{2}}$. Let $\mu_0(T) \geq \mu_1(T) \geq ...$ be the sequence of the eigenvalues of $|T|$; repeated according to their multiplicity. Denoted $\sigma_N(T) = \sum_{j=1}^{N} \mu_j(T)$ we define $\mathcal{L}^{(1,\infty)}(H) = \{T \in \mathbf{K}(H), \sigma_N(T) = O(logN)\}$ endowed with the norm $\| T \|_{1,\infty} = sup_{N\geq 2} \frac{\sigma_N(T)}{logN}$. $\mathcal{L}^{(1,\infty)}(H)$ is a two-side ideal of $\mathcal{L}(H)$. Then, consider a linear form ω on $C_b(1,\infty)$ with $\omega \geq 0$, $\omega(1) = 1$ and $\omega(f) = 0$ if $\lim_{x\mapsto\infty} f(x) = 0$. Given a bounded sequence $a = (a_n)_{n\geq 1}$ we construct the function $f_a = \sum_{n\geq 1} a_n \chi_{[n-1,n)} \in L^\infty(\mathbb{R}^+)$ and define the ω-limit $\lim_\omega a_N = \omega(Mf_a)$ where, for $g \in L^\infty(\mathbb{R}^+)$, $Mg(t) := \frac{1}{logt} \int_1^t \frac{f(s)}{s} ds$ is the Cesáro mean of g. In the case of convergent sequence the ω-limit coincides with the usual limit.

Definition 4.1. *Let $T \in \mathcal{L}^{(1,\infty)}(H)$be a positive self-adjoint operator. We define Dixmier trace of T as*

$$Tr_\omega(T) = \lim_\omega \tfrac{1}{logN} \sum_{n=0}^{N} \mu_n(T).$$

Dixmier trace can be extended to a linear map on $T \in \mathcal{L}^{(1,\infty)}(H)$ also denoted by Tr_ω.

Proposition 4.2. *[6] The Dixmier trace Tr_ω has a canonical extension to a trace on the algebra of Heisenberg pseudodifferential operators of arbitrary order. It is given globally by the equality*

$$Tr_\omega(T) = \tfrac{1}{p+2q} \int_M c(x),$$

where c(x) is defined in (3.7) and T is an Heisenberg pseudodifferential operator of order $-(p+2q)$

Definition 4.3. *[10] Let $\mathcal{L}^2$ denote the class of Hilbert-Schmidt operators on the Hilbert space $L^2(\mathbb{R}^n)$. Let*

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the Laplacian on $\mathbb{R}^n$. *Let* $n \in \mathbb{N}$. *A bounded operator* $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is called Laplacian modulated if*

$$\| A \|_{mod} = \sup_{t>0} t^{1/2} \left\| A \left(1 + t(1-\Delta)^{-n/2}\right)^{-1} \right\|_{\mathcal{L}^2} < \infty$$

Definition 4.4. *[13] The map res, from the set of compactly based Laplacian-modulated operator to the quotient* ℓ^∞ / c_0, *is defined for any compactly based Laplacian-modulated operator* T *by*

$$\operatorname{res}(T) := \left[\left(\int_{\|\xi\|' \leq j^{1/(p+q)}} \int_{\mathbb{R}^{p+q}} p_T(x,\xi) \mathrm{d}x \ \mathrm{d}\xi \right)_{j \in \mathbb{N}} \right]$$

where p_T *denotes the symbol associated with* T *and* $[\cdot]$ *denotes the equivalence class in* ℓ^∞ / c_0.

Proposition 4.5. *For any classical and compactly based Heisenberg pseudo-differential operator* A *of order* $-(p+2q)$ *one has*

$$\operatorname{res}(A) = \frac{p+2q}{q} \operatorname{Tr}_\omega(A) + C,$$

where $\operatorname{res}(A)$ is defined in (4.4) and $C = \int_{\mathbb{R}^{p+q}} \int_{\|\xi\|'=1} a_{-(p+2q)}(x,\xi) dx d\xi$

Proof. Let us denote by $a_{-(p+2q)}$ the principal symbol of the operator A . By the definition, $\operatorname{res}(A)$ depends only on the symbol $a_{-(p+2q)}$, and is determined by the equivalence class in ℓ^∞ / c_0 of the sequence

$$\left(\int_{\|\xi\|' \leq j^{1/(p+q)}} \int_{\mathbb{R}^{p+q}} a_{-(p+2q)}(x,\xi) \mathrm{d}x \ \mathrm{d}\xi \right)_{j \in \mathbb{N}}.$$

Since $a_{-(p+2q)}$ is homogeneous of order $-(p+2q)$ and is compact in its first variable one has

$$\begin{aligned}
&\int_{\|\xi\|' \leq j^{1/(p+q)}} \int_{\mathbb{R}^{p+q}} a_{-(p+2q)}(x,\xi) \mathrm{d}x \ \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{p+q}} \int_{1 < \|\xi\|' \leq j^{1/(p+2q)}} \| \xi \|'^{-(p+2q)} \ a_{-(p+2q)} \left(x, \frac{\xi}{\| \xi \|'} \right) \mathrm{d}\xi \mathrm{d}x + C \\
&= \int_{\mathbb{R}^{p+q}} \int_{\|\theta\|'=1} a_{-(p+2q)}(x,\theta) \mathrm{d}\theta \mathrm{d}x \int_1^{j^{1/(p+q)}} r^{-(p+2q)} r^{(p+q)-1} \ \mathrm{d}r + C \\
&= \int_{\mathbb{R}^{p+q}} \int_{\|\theta\|'=1} a_{-(p+2q)}(x,\theta) \mathrm{d}\theta \mathrm{d}x \int_1^{j^{1/(p+q)}} r^{-q-1} \mathrm{d}r + C \\
&= \frac{1}{q} \left(1 - j^{\frac{-q}{p+q}} \right) \int_{\mathbb{R}^{p+q}} \int_{\|\theta\|'=1} a_{-(p+2q)}(x,\theta) \mathrm{d}\theta \mathrm{d}x + C
\end{aligned}$$

with C a constant independent of j. As a consequence one infers that

$$\begin{aligned}\mathrm{res}(A) &= \left[\left(\frac{1}{q}\left(1-j^{\frac{-q}{p+q}}\right)\int_{\|\xi\|'\leq j^{1/(p+2q)}}\int_{\mathbb{R}^{p+q}} a_{-(p+2q)}(x,\xi)\mathrm{d}x\ \mathrm{d}\xi\right)_{j\in\mathbb{N}}\right]+C\\ &= \left[\left(\frac{1}{q}\int_{\mathbb{R}^{p+q}}\int_{\|\theta\|'=1} a_{-(p+2q)}(x,\theta)\mathrm{d}\theta\mathrm{d}x\right)\right]+C\\ &= \frac{p+2q}{q}\,\mathrm{Tr}_{\omega}(A)+C\end{aligned}$$

with the identification mentioned before the statement of the proposition. □

References

[1] M. Adler . *On a trace functional for formal pseudo-differential operators and the symplectic structure of Korteweg-de Vries type equations, Inventiones Math. 50 (1979), 219-248.*

[2] R. Beals, P. Greiner. *Calculus on Heisenberg manifolds, Annals of Math. Studies 119 (1985).*

[3] A. Connes. *Noncommutative Geometry. Academic Press, 1995.*

[4] A. Connes. *The action functional in Noncommutative Geometry. CMP 117 (1988)pp. 673-683.*

[5] A. Connes. *Géométrie du point de vue spectrale et brisure spontannée de symétrie. Séminaire de Bourbaki, Juin 1996.*

[6] A. Connes, H. Moscovici, *The local index formula in noncommutaive geometry.* Geom. Funct. Anal. 5 (1995), no. 2, 345-388.

[7] C. M. Ducourtioux. *Weighted traces on pseud-differential operators and associated determinants. PhD Thesis, Université Blaise Pascal, 2001.*

[8] Guillemin V., *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues, Adv. Math.55 (1985),131-160.*

[9] L. Hormander, *"The Analysis of Linear Partial Differential Operators III," Grundlehrender mathematishen Wissenschaften 274, Springer-Verlag, Berlin, Heidelberg, 1994.*

[10] N. Kalton, S. Lord, D. Potapov, F. Sukochev, *Traces of compact operators and the noncommutative residue, Adv. Math. 235,1-55, 2013.*

[11] C. Kassel. *Le résidu non commutatif(d'après M. Wodzicki). Astérisque, (177-178): EXP. No. 708, 199-229, 1989. Séminaire Bourbaki, Vol. 1989/89.*

[12] P. Libermann - *Formes différentielles sur une variété symplectique, C.R. Acad. Sci. Paris 234 (1952), 395-397.*

[13] S. Lord, F. Sukochev, D. Zanin, *Singular traces, theory and applications, Studies in mathematics 46, De Gruyter, 2013*

[14] S. Minakshisundaram and A. Pleijel. *Some properties of the eigenfunctions of the laplace-operator on riemannian manifolds. Canadian Journal of Mathematics, 1 :242?256, 1949.*

[15] R. Ponge, *Noncommutative residue for Heisenberg manifolds. Applications in CR and contact geometry.*

[16] R. Ponge, *Calcul hypoelliptique sur les variétés de Heisenberg, résidu non commutatif et géométrie pseudo-hermitienne.* PhD thesis, Université Paris-Sud, Orsay, 2000.

[17] E. Schrohe *Noncommutative Residues, Dixmier's Trace, and Heat Trace Expansions on Manifolds with Boundary. In Geometric aspects of partial differential equations (Roskilde, 1998), pages 161-186. Amer. Math. Soc., Providence, RI,1997.*

[18] R. T. Seely, *Complex powers of an elliptic operator in Proc. Sympos. Pure Math. 10, 288-307, Amer. Math. Soc.(1968).*

[19] S. Vassout. *Feuilletages et Résidu non commutatif longitudinal. PhD Thesis, Université Pierre et Marie Currie-Paris VI, 2001.*

[20] M. Wodzicki. *Noncommutative residue. I. Fundamentals. K-theory, arithmeticand geometry (Moscow, 1984-1986), 320-399, Lecture Notes in Math., 1289, Springer, Berlin-New York, 1987.*