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Null controllability of a nonlinear age structured model for a two-sex population

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Abstract : This paper is devoted to study the null controllability properties of a nonlinear age and two-sex population dynamics structured model without spatial structure. Here, the nonlinearity and the couplage are at birth level. In this work we consider two cases of null controllability problem: The first problem is related to the extinction of male and female subpopulation density. The second case concerns the null controllability of male or female subpopulation individuals. In both cases, if A is the maximal age, a time interval of duration A after the extinction of males or females, one must get the total extinction of the population. Our method uses first an observability inequality related to the adjoint of an auxiliary system, a null controllability of the linear auxiliary system and after the Kakutani's fixed point theorem.

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1 Introduction

In this article we will study the null controllability of a nonlinear age structured model for a two - sex population. This system models for example the dynamic of population of mosquitoes.

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Consequently, this study can be used to control the size of male and female mosquitoes in a given area and therefore contribute to the fight against the spread of malaria.

In the theoretical framework, very few authors have studied control problems of two-sex structured population dynamics model.

The control problems of coupled systems of population dynamics models take an intense interest and are widely investigated in many papers. Among them, we can cite [1], [9], [16],[14], [11] and the references therein. In fact, in [1] the authors studied a coupled reaction-diffusion equations describing interaction between a prey population and a predator one. The goal of the above work is to look for a suitable control supported on a small spatial subdomain which guarantees the stabilization of the predator population to zero. In [16], the objective was different. More precisely, the authors consider an age-dependent prey predator system and they prove the existence and uniqueness of an optimal control (called also "optimal effort") which gives the maximal harvest via the study of the optimal harvesting problem associated to their coupled model. In [8] He and Ainseba study the null controllability of a butterfly population by acting on eggs, larvas and female moths in a small age interval.

In [9], the authors analyze the growth of a two-sex population with a fixed age-specific sex ratio without diffusion. The model is intended to give an insight into the dynamics of a population where the mating process takes place at random choice and the proportion between females and males is not influenced by environmental or social factors, but only depends on a differential mortality or on a possible transition from one sex to the other (e.g. in sequential hermaphrodite species).

Simporé and Traoré study in [11] the null controllability of a nonlinear age, space and two-sex structured population dynamics model. They first study an approximate null controllability result for an auxiliary cascade system and prove the null controllability of the nonlinear system by means of Schauder's fixed point theorem.

In [14], A. Traoré, O. S. Sougué, Y. Simporé and O. Traoré study the null controllability of the model presented in [11] without space. They first establish an observability inequality of the adjoint system which serves to show the approximate controllability; then the null controlability using Kakutani's fixed point theorem.

In [14], the control of males and females act respectively on $\Theta_1 = (a_1, a_2) \times (0, T)$ and $\Theta_2 = (b_1, b_2) \times (0, T)$ and with the condition $(a_1, a_2) \subset (b_1, b_2)$.

Our aim in this article is to study the null controllability of the model presented in [14] with the more general condition $(a_1, a_2) \cap (b_1, b_2) \neq \emptyset$. Thus we improve the results presented in [14]and [11].

2 Model and main results

In this paper, we study the null controllability of a nonlinear coupled system describing the dynamics of two-sex structured population. Let (m, f) be the solution of the following system :

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = \chi_{\Theta_1} v_m & \text{in } Q, \\ \partial_t f + \partial_a f + \mu_f f = \chi_{\Theta_2} v_f & \text{in } Q, \\ m(a,0) = m_0(a) \quad f(a,0) = f_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,M) f(a,t)da & \text{in } Q_T, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,M) f(a,t)da & \text{in } Q_T, \\ M = \displaystyle\int_0^A \lambda(a) m(a,t)da & \text{in } Q_T, \end{cases} \tag{2.1}$$

where T is a positive number, $Q = (0, A) \times (0, T)$, $\Theta = (0, a_2) \times (0, T)$, $\Theta_1 = (a_1, a_2) \times (0, T)$ and $\Theta_2 = (b_1, b_2) \times (0, T)$. Here $0 \leq a_1 < a_2 \leq A$, $0 \leq b_1 < b_2 \leq A$, $Q_A = (0, A) \times \{0\}$ and $Q_T = \{0\} \times (0, T)$.
We denote the density of males and females of age a at time t respectively by $m(a, t)$ and $f(a, t)$. Moreover, μ_m and μ_f denote respectively the natural mortality rate of males and females. The control functions are v_m and v_f and depend on a and t. In addition χ_{Θ_1} and χ_{Θ_2} are the characteristic functions of the support of the control v_m and v_f respectively.
We have denoted by β the positive function describing the fertility rate that depends on a and also on

$$M = \int_0^A \lambda(a) m(a,t)da,$$

where λ is the fertility function of the male individuals. Thus the densities of newborn male and female individuals at time t are given respectively by $m(0, t) = (1 - \gamma)N(t)$ and $f(0, t) = \gamma N(t)$ where

$$N(t) = \int_0^A \beta(a,M) f(a,t)da.$$

We assume that the fertility rate β, λ and the mortality rate μ_f, μ_m satisfy the demographic properties :

$$(H_1) \begin{cases} \mu_m(a) \geq 0, \quad \mu_f(a) \geq 0 \text{ a.e } a \in (0, A) \\ \mu_m \in L^1_{loc}(0, A), \quad \mu_f \in L^1_{loc}(0, A) \\ \displaystyle\int_0^A \mu_m(a)da = +\infty, \quad \int_0^A \mu_f(a)da = +\infty \end{cases}$$

$$(H_2) \begin{cases} \beta(a,p) \in C([0, A] \times \mathbb{R}) \\ \beta(a,p) \geq 0 \text{ for every } (a,p) \in [0, A] \times \mathbb{R}. \end{cases}$$

We further assume that the birth function β and the fertility function λ verify the following hypotheses:

$$(H_3)\begin{cases} \text{there exists } b \in (0,A) \text{ such that } \beta(a,p)=0,\ \forall (a,p)\in(0,b)\times\mathbb{R},\\ \text{there exists a constant } \alpha_+>0 \text{ such that } 0\le\beta\le\alpha_+,\ \forall(a,p)\in(0,A)\times\mathbb{R},\\ \beta(a,0)=0,\ \forall\ a\in(0,A).\end{cases}$$

$$(H_4)\begin{cases} (i) \quad \lambda\in C^1([0,A])\\ (ii) \quad \lambda(a)\ge 0 \text{ for every } a\in[0,A],\\ (iii) \quad \lambda\mu_m\in L^1(0,A).\end{cases}$$

To illustrate the hypothesis (H_4), consider the following classic examples of demographic functions:

$$\lambda(a)=\begin{cases}\exp\{-\frac{1}{A-a}\} & \text{if} \quad a\in[0;A[\\ 0 & \text{if} \quad a=A\end{cases} \qquad \text{and} \qquad \mu_m(a)=\frac{1}{A-a}.$$

It is clear taht λ satisfies $(H_5)-(i)-(ii)$ and μ_m satisfies (H_1). And we have

$$\lim_{a\to A^-}\lambda\mu_m(a)=\lim_{a\to A^-}\frac{1}{A-a}\exp\{-\frac{1}{A-a}\}=0$$

Then $\lambda\mu_m$ is continuous on $[0;A)$ and extendable by continuity in A and $[0;A]$ is a compact of $\mathbb{R}$, so $\lambda\mu_m$ is integrable, therefore $\lambda\mu_m\in L_1(0;A)$.

Remark 2.1. *In population dynamics, we can assume that there are ages a_1 and a_2 with $a_1<a_2$ in $[0,A]$ such that $\lambda(a)=0$ for $a<a_1$ and $a>a_2$, (biologically, this means that very young and very old individuals are not fertile) which ensures that $\lambda\mu_m\in L^1(0,A)$.*
Indeed, if λ and μ_m are positive and $\lambda\in C^1([0;A])$ and $\mu_m\in L^1_{loc}(0,A)$; one has

$$\int_0^A\lambda(a)\mu_m(a)(a)=\tilde{\lambda}\int_{a_1}^{a_2}\mu_m(a)da<+\infty$$

where $\tilde{\lambda}=\max\limits_{a\in[a_1;a_2]}\lambda$, so $\lambda\mu_m\in L^1(0,A)$ because $\mu_m\in L^1_{loc}(0,A)$

Remark 2.2. *The assumption $\beta(a,0)=0$ for $a\in(0,A)$ means that, the birth rate is zero if there are no fertile male individuals.*

We have the following theorem:

Theorem 2.1. *Let us assume that the assumptions $(H_1)-(H_2)-(H_3)-(H_4)$ hold true. If $(0,b)\cap(a_1,a_2)\cap(b_1,b_2)\neq\emptyset$, for every time $T>\max\{a_1,b_1\}+\max\{A-a_2,A-b_2\}$ and for every $(m_0,f_0)\in\left(L^2(Q_A)\right)^2$, there exists $(v_m,v_f)\in L^2(\Theta_1)\times L^2(\Theta_2)$ such that the associated solution (m,f) of system (2.1) verifies:*

$$m(a,T)=f(a,T)=0 \quad a.e \quad a\in(0,A). \tag{2.2}$$

Remark 2.3. *It is certainly possible to achieve the total extinction of the population under the assumptions: $(a_1,a_2)\cap(b_1,b_2)=\emptyset$, $a_1\geqslant b$ or $b_1\geqslant b$. But under these conditions our method does not allow us to establish the observability inequality (4.13).*

Theorem 2.2. *Let us assume that the assumptions $(H_1)-(H_2)-(H_3)-(H_4)$ hold true. We have :*

(1) let $v_f = 0$. For any $\varrho > 0$, for every time $T > A - a_2$ and for every $(m_0, f_0) \in \left(L^2(Q_A)\right)^2$, there exists a control $v_m \in L^2(\Theta)$ such that the associated solution (m, f) of system (2.1) verifies:

$$m(a,T) = 0 \quad a.e\, a \in (\varrho, A) \tag{2.3}$$

where $\Theta = (0, a_2) \times (0, T)$.

(2) let $v_m = 0$. For every time $T > a_1 + A - a_2$ and for every $(m_0, f_0) \in \left(L^2(Q_A)\right)^2$, there exists a control $v_f \in L^2(\Theta_1)$ such that the associated solution (m, f) of system (2.1) verifies:

$$f(a,T) = 0 \quad a.e\, a \in (0, A). \tag{2.4}$$

Moreover, if we obtain $f(a,T) = 0$; one has at time $T + A$,

$$m(a, T + A) = 0 \quad a.e \quad a \in (0, A), \tag{2.5}$$

$$f(a, T + A) = 0 \quad a.e \quad a \in (0, A). \tag{2.6}$$

Remark 2.4. *It should be noted here that the control of males acts on the age interval $(0; a_2)$. Indeed if the lower limit of the controlled age interval is $a_1 > 0$, the male individuals born between $T - a_1$ and T will not be old enough to belong to the controlled age class which will be $(a_1; a_2)$. Therefore, it would be impossible to achieve the total extinction of males. We will then give the mathematical justification in the part devoted to the proof of Theorem 2.2*

Remark 2.5. *The first condition of (H_3) is not necessary for the Theorem 2.2-(1).*

We use the technique of [11] and [10] combining final-state observability estimates with the use of characteristics to establish the observability inequalities necessary for the null controllability property of the auxiliary systems. Roughly, in our method we first study the null controllability result for an auxiliary cascade system. Afterwards, we prove the null controllability result for the system (2.1) by means of Kakutani's fixed point theorem.
The remainder of this paper is as follows: in Section 2 we describe the model and give the main results. Then we study the existence and uniqueness of a positive solution for the model in Section 3. Section 4 is devoted to the proofs of Theorem 2.1 and Theorem 2.2 respectively.

3 Well posedness result

In this section, we study the existence of positive solution of the model. For this, we assume that the so-called demographic conditions (H_1), (H_2), (H_3) and (H_4) are verified. Moreover, here, we suppose that

$$(H_5)\begin{cases} (i) \quad \beta(a,p) = \beta_1(a)\beta_2(p) \text{ for all } (a,p) \in (0,A) \times \mathbb{R}, \\ (ii) \quad \text{there exists } C > 0 \text{ such that } |\beta_2(p) - \beta_2(q)| \leq C|p - q| \text{ for all } p, q \in \mathbb{R}, \\ (iii) \quad \beta_1 \mu_f \in L^1(0, A) \end{cases}$$

holds true.
Thus, we have the following result.

Theorem 3.1.
Assume that $(H_1)-(H_5)$ hold. For every $(m_0,f_0)\in(L^2(0,A))^2$ and $(v_m,v_f)\in(L^2(Q))^2$, the system (2.1) admits a unique solution $(m,f)\in(L^2((0,A)\times(0,T)))^2$ and the following estimates occur:

$$\begin{aligned}&||m||_{L^2((0,A)\times(0,T))}\leq K\big(||f_0||_{L^2(0,T)}+||m_0||_{L^2(0,T)}+||v_m||_{L^2(Q)}+||v_f||_{L^2(Q)}\big),\\ &||f||_{L^2((0,A)\times(0,T))}\leq C\big(||m_0||_{L^2(0,T)}+||v_f||_{L^2(Q)}\big)\end{aligned}\tag{3.1}$$

where K and C are positive constants.
Moreover, suppose that

$$m_0,f_0\geq 0\ a.e\,(0,A)\ and\ v_m,v_f\geq 0\ a.e\,Q;$$

then (m,f) is also positive.

Proof of Theorem 3.1: Let p be fixed in $L^2(0,T)$, h and h' be fixed in $L^2(Q)$ and consider the following system

$$\begin{cases}\partial_t m+\partial_a m+\mu_m m=h & \text{in } Q,\\ \partial_t f+\partial_a f+\mu_f f=h' & \text{in } Q,\\ m(a,0)=m_0(a)\quad f(a,0)=f_0(a) & \text{in } Q_A,\\ m(0,t)=(1-\gamma)\displaystyle\int_0^A\beta\left(a,\int_0^A\lambda(a)p(a,t)\right)f(a,t)da & \text{in } Q_T,\\ f(0,t)=\gamma\displaystyle\int_0^A\beta\left(a,\int_0^A\lambda(a)p(a,t)\right)f(a,t)da & \text{in } Q_T.\end{cases}\tag{3.2}$$

For every $f_0\in L^2(0,A)$ and $h'\in L^2(Q)$, the following system

$$\begin{cases}\partial_t f+\partial_a f+\mu_f f=h' & \text{in } Q,\\ f(a,0)=f_0(a) & \text{in } Q_A,\\ f(0,t)=\gamma\displaystyle\int_0^A\beta\left(a,\int_0^A\lambda(a)p(a,t)\right)f(a,t)da & \text{in } Q_T,\end{cases}\tag{3.3}$$

admits a unique positive solution in $L^2(Q)$, (see [3],[15]) and one has

$$\|f\|^2_{L^2(Q)}\leqslant C\left(\|f_0\|^2_{L^2(0,A)}+\|h'\|^2_{L^2(Q)}\right),\tag{3.4}$$

where C is a positive constant and independent of p because $\beta\in L^\infty((0,T)\times(0,A))$.

Now, f and h' are being known, the system

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = h & \text{in } Q, \\ m(a,0) = m_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta\left(a, \int_0^A \lambda(a)p(a,t)\right) f(a,t)da & \text{in } Q_T, \end{cases} \tag{3.5}$$

admits a unique positive system in $L^2(Q)$ and we have the following estimation

$$\|m\|^2_{L^2(Q)} \leqslant K\left(\|f_0\|^2_{L^2(0,A)} + \|m_0\|^2_{L^2(0,A)} + \|h\|^2_{L^2(Q)} + \|h'\|^2_{L^2(Q)}\right),$$

where K is a positive constant and independent of p because $\beta \in L^\infty((0,T)\times(0,A))$.

Let define $\Phi\ :\ L^2_+(Q) \longrightarrow L^2_+(Q)$, $\Phi(p) = m(p)$ where $m(p)$ is the unique solution of the system (3.5).

For any $p\ ,\ q \in L^2_+(Q)$, we set

$$B_1(a,t) = \int_0^A \lambda(a)p(t,a)da \quad \text{and} \quad B_2(a,t) = \int_0^A \lambda(a)q(t,a) \quad \text{a.e.} \quad t \in (0,A)\times(0,T),$$

and $w = (m(p) - m(q))e^{-\gamma_0 t}$ where γ_0 is a positive parameter that will be choosed later; w is solution of

$$\begin{cases} \partial_t w + \partial_a w + (\gamma_0 + \mu_m)w = 0 & \text{in } Q, \\ w(a,0) = 0 & \text{in } Q_A, \\ w(0,t) = (1-\gamma)e^{-\gamma_0 t}\times & \\ \qquad \displaystyle\int_0^A \left[\beta_2(B_1) - \beta_2(B_2)\right]\beta_1(a)f(p) + (f(p) - f(q))\beta_2(B_2)\beta_1(a)da & \text{in } Q_T. \end{cases} \tag{3.6}$$

Multiplying (3.6) bye w and integrating over $(0,A)\times(0,t)$, and using Young's inequality we get

$$\begin{aligned}\frac{1}{2}\|w(t)\|^2_{L^2(0,A)} + \int_0^t\int_0^A(\gamma_0+\mu_m)w^2(s,a)dads &\leqslant \int_0^t\left(\int_0^A|\beta_2(B_1)-\beta_2(B_2)|\,\beta_1(a)f(p)da\right)^2 ds\\
&+\int_0^t\left(\int_0^A(f(p)-f(q))\,\beta_2(B_2)\beta_1(a)da\right)^2 ds\\
&\leqslant C^2\|\lambda\|^2_\infty\int_0^t\left(\left|\int_0^A p(s,a)da-\int_0^A q(s,a)da\right|\right)^2\left(\int_0^A\beta_1(a)f(p(s))da\right)^2 ds\\
&+\int_0^t\left(\int_0^A(f(p)-f(q))\,\beta_2(B_2)\beta_1(a)da\right)^2 ds\\
&\leqslant C^2A\|\lambda\|^2_\infty\int_0^t\int_0^A|p(s)-q(s)|^2\left(\int_0^A\beta_1(a)f(p(s))da\right)^2 ds\\
&+\|\beta_1\|^2_\infty\|\beta_2\|^2_\infty A\int_0^t\int_0^A|f(p)-f(q)|^2 ds.\end{aligned}$$

Hence for every $\gamma_0>0$, there is a constant $C=\max\left\{2C^2A\|\lambda\|^2_\infty\,;2\|\beta_1\|^2_\infty\|\beta_2\|^2_\infty A\right\}$ such that

$$\|w(t)\|^2_{L^2(0,A)}\leqslant C\left(\int_0^t\int_0^A|p(s)-q(s)|^2\left(\int_0^A\beta_1(a)f(p(s))da\right)^2 ds+\int_0^t\int_0^A|f(p(s))-f(q(s))|^2 ds\right) \tag{3.7}$$

Now set $F=(f(p)-f(q))e^{-\delta t}$ where δ is a positive parameter that will be choosed later. Then, F solves the following auxiliary system

$$\begin{cases}\partial_t F+\partial_a F+(\delta+\mu_f)F=0 & \text{in } Q,\\ F(a,0)=0 & \text{in } Q_A,\\ w(0,t)=\gamma\int_0^A e^{-\gamma_0 t}\left[\beta_2(H_1)-\beta_2(H_2)\right]\beta_1(a)f(p)+F(a,t)\beta_2(H_2)\beta_1(a)da & \text{in } Q_T.\end{cases} \tag{3.8}$$

Similarly as above, we have

$$\delta\int_0^t\int_0^A F(a,s)^2ds\leqslant C\left(\int_0^t\int_0^A|p(s)-q(s)|^2\left(\int_0^A\beta_1(a)f(p(s))da\right)^2 ds+\int_0^t\int_0^A|F(a,s)|^2 ds\right).$$

Hence, there is a positive constant C' such that

$$\int_0^t\int_0^A F(a,s)^2ds\leqslant C'\int_0^t\int_0^A|p(s)-q(s)|^2\left(\int_0^A\beta_1(a)f(p(s))da\right)^2 ds. \tag{3.9}$$

Setting $Y(t)=\int_0^A \beta_1(a)f(p)da$ a.e in $(0,T)$, Y solves the following system

$$\begin{cases} \partial_t Y = \int_0^A \beta_1(a)h'(a,t)da + \int_0^A \beta_1'(a)f(a,t)da - \int_0^A \mu_f(a)\beta_1(a)f(a,t)da & \text{in } (0,T), \\ Y(0) = \int_0^A \beta_1(a)f_0(a)da \end{cases} \tag{3.10}$$

Multiplying (3.10) by Y, integrating over $(0,t)$ and using Young's inequality we get

$$\begin{aligned} Y^2(t) &\leqslant Y^2(0) + \int_0^t Y^2(s)ds + \int_0^t \left(\int_0^A \beta_1(a)h'(a,s)da + \int_0^A \beta_1'(a)f(a,s)da + \int_0^A \mu_f(a)\beta_1(a)f(a,s)da\right)^2 ds \\ &\leqslant Y^2(0) + \int_0^t Y^2(s)ds + 3\int_0^t \left(\int_0^A \beta_1(a)h'(a,s)da\right)^2 ds + 3\int_0^t \left(\int_0^A \beta_1'(a)f(a,s)da\right)^2 ds \\ &\quad + 3\int_0^t \left(\int_0^A \beta_1(a)\mu_f(a)f(a,s)da\right)^2 ds. \end{aligned}$$

So,

$$\begin{aligned} Y^2(t) &\leqslant \left(\int_0^A \beta_1(a)f_0(a)da\right)^2 + \int_0^T Y^2(t)dt + 3\int_0^T \left(\int_0^A \beta_1(a)h'(a,t)da\right)^2 dt + 3\int_0^T \left(\int_0^A \beta_1'(a)f(a,t)da\right)^2 dt \\ &\quad + 3\int_0^T \left(\int_0^A \beta_1(a)\mu_f(a)f(a,t)da\right)^2 dt. \end{aligned} \tag{3.11}$$

Let us set $\tilde{f} = e^{-\lambda_0 t}f$. Then, from (3.3) $\tilde{f}$ satisfies the following system

$$\begin{cases} \partial_t \tilde{f} + \partial_a \tilde{f} + (\lambda_0 + \mu_f)\tilde{f} = e^{-\lambda_0 t}h' & \text{in } Q, \\ \tilde{f}(a,0) = f_0(a) & \text{in } Q_A, \\ \tilde{f}(0,t) = \gamma\int_0^A \beta\left(a, \int_0^A \lambda(a)p(a,t)\right)\tilde{f}(a,t)da & \text{in } Q_T. \end{cases} \tag{3.12}$$

Multiplying the first equation of (3.12) by $\tilde{f}$, integrating on Q and using Young's inequality we get

$$\int_0^T\int_0^A (\lambda_0 + \mu_f(a))\tilde{f}^2(a,t)dadt \leq \frac{1}{2}||f_0||^2_{L^2(0,A)} + \frac{1}{2}||h'||^2_{L^2(Q)} + \frac{1}{2}||\tilde{f}||^2_{L^2(Q)} + \frac{1}{2}\int_0^T \tilde{f}^2(0,t)dt.$$

Using Cauchy Schwarz's inequality and choosing $\lambda_0 = \frac{3}{2} + \alpha_+^2$, we obtain

$$\int_0^T\int_0^A \mu_f(a)\tilde{f}^2(a,t)dadt \leq \frac{1}{2}\left(||f_0||^2_{L^2(0,A)} + ||h'||^2_{L^2(Q)}\right).$$

So,

$$\int_0^T \int_0^A \mu_f(a) f^2(a,t)dadt \leq \frac{e^{(3+2||\beta||_\infty^2)T}}{2}\Big(||f_0||^2_{L^2(0,A)} + ||h'||^2_{L^2(Q)}\Big). \tag{3.13}$$

Using (3.11), (3.13) and against Young's inequality we have

$$\begin{aligned} Y^2(t) \leqslant & \|\beta_1\|^2_\infty A \|f_0\|^2_{L^2(0,A)} + \|\beta_1\|^2_\infty A \|f\|^2_{L^2(Q)} + 3\|\beta_1\|^2_\infty A \|h'\|^2_{L^2(Q)} \\ & + 3\|\beta_1'\|^2_\infty A \|f\|^2_{L^2(Q)} + 3C||\beta_1||_\infty \|\beta_1\mu_f\|_{L^1(0,A)} ||f_0||^2_{L^2(0,A)} \\ & + 3C||\beta_1||_\infty \|\beta_1\mu_f\|_{L^1(0,A)} ||h'||^2_{L^2(Q)}. \end{aligned}$$

From (3.4), we have just proved the existence of a positive constant C such that ,

$$Y^2(t) \leqslant C\left(\|f_0\|^2_{L^2(0,A)} + \|h'\|^2_{L^2(0,A)}\right) \tag{3.14}$$

The estimate (3.14) means also that $Y \in L^\infty(0,T)$.
Combining (3.7), (3.9) and (3.14), we get the following estimate

$$\|(\Phi(p) - \Phi(q))(t)\|^2_{L^2(0,A)} \leqslant \sigma \int_0^t \|p(s) - q(s)\|^2_{L^2(0,A)} ds, \tag{3.15}$$

where σ is a positive constant.
Let us define the metric d on $L^2_+(Q)$ by setting

$$d(h_1,h_2) = \left(\int_0^T \|(h_1-h_2)(t)\|^2_{L^2((0,A)} \exp\{-2\sigma t\}dt\right)^{\frac{1}{2}}, \quad \text{for } h_1,h_2 \in L^2_+(Q).$$

We have

$$d(\Phi(p),\Phi(q))^2 = \int_0^T \|(\Phi(p){-}\Phi(q))(t)\|^2_{L^2((0,A)} \exp\{-2\sigma t\}dt \leqslant \sigma \int_0^T \exp\{-2\sigma t\} \int_0^t \|(p{-}q)(s)\|^2_{L^2((0,A)} dsdt$$

Using Fubbini's theorem, we conclude that

$$\begin{aligned} d(\Phi(p),\Phi(q))^2 = \int_0^T \|(\Phi(p) - \Phi(q))(t)\|^2_{L^2((0,A)} \exp\{-2\sigma t\}dt & \leqslant \int_0^T \|(p-q)(s)\|^2_{L^2((0,A)} \times \int_s^T \sigma e^{-2\sigma t} dtds \\ & \leqslant \frac{1}{2} d(p,q)^2. \end{aligned}$$

Then, Φ is a contraction on the complete metric space $L^2_+(Q)$ into itself. Using Banach's fixed point theorem, we conclude the existence of a unique fixed point m. Moreover, m is nonnegative. Hence, the unique couple (m,f) is the unique solution to our problem (2.1).
The reader can consult [11], [14] □

4 Null controllability results

For the sequel, the hypothesis (H_5) is not necessary. As a consequence, the uniqueness and the positivity of the solution of system (2.1) are not garanteed.
We first establish an observability inequality to show the controllability of a linear system. Then, by a fixed point method we show the controllability of the model.

4.1 Null controllability of an auxiliary coupled system

This section is devoted to the study of an auxiliary system obtained from the system (2.1). Let p be a $L^2(Q_T)$ function, we define the auxiliary system given by:

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = \chi_{\Theta_1} v_m & \text{in } Q, \\ \partial_t f + \partial_a f + \mu_f f = \chi_{\Theta_2} v_f & \text{in } Q, \\ m(a,0) = m_0(a) \quad f(a,0) = f_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,p) f(a,t) da & \text{in } Q_T, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,p) f(a,t) da & \text{in } Q_T. \end{cases} \tag{4.1}$$

Let p be fixed in $L^2(Q_T)$, for $(m_0, f_0) \in \big(L^2(Q_A)\big)^2$ and $(v_m, v_f) \in L^2(\Theta_1) \times L^2(\Theta_2)$ the system (4.1) admits a unique solution $(m,f) \in \big(L^2(Q)\big)^2$, see Section 3.
The adjoint system of (4.1) is given by:

$$\begin{cases} -\partial_t n - \partial_a n + \mu_m n = 0 & \text{in } Q, \\ -\partial_t l - \partial_a l + \mu_f l = (1-\gamma)\beta(a,p) n(0,t) + \gamma \beta(a,p) l(0,t) & \text{in } Q, \\ n(a,T) = n_T(a), \quad l(a,T) = l_T(a) & \text{in } Q_A, \\ n(A,t) = 0, \quad l(A,t) = 0 & \text{in } Q_T. \end{cases} \tag{4.2}$$

Lemma 4.1. *For every $(n_T, l_T) \in (L^2(Q_A))^2$, under the assumptions (H_1) and (H_2), the coupled system (4.2) admits a unique solution (n,l). Moreover integrating along the characteristic lines, the solution (n,l) of (4.2) is as follows:*

$$n(a,t) = \begin{cases} \dfrac{\pi_1(a+T-t)}{\pi_1(a)} n_T(a+T-t) \quad \text{if} \quad T-t \le A-a, \\ \\ 0 \quad \text{if} \quad A-a < T-t \end{cases} \tag{4.3}$$

and

$$l(a,t) = \begin{cases} \dfrac{\pi_2(a+T-t)}{\pi_2(a)} l_T(a+T-t) \\ + \displaystyle\int_t^T \dfrac{\pi_2(a+s-t)}{\pi_2(a)} \beta(a+s-t, p(s))\big((1-\gamma)n(0,s) + \gamma l(0,s)\big) ds \\ \text{if} \quad T-t \le A-a, \\ \displaystyle\int_t^{t+A-a} \dfrac{\pi_2(a+s-t)}{\pi_2(a)} \beta(a+s-t, p(s))\big((1-\gamma)n(0,s) + \gamma l(0,s)\big) ds \\ \text{if} \quad A-a < T-t, \end{cases} \tag{4.4}$$

where $\pi_1(a) = e^{-\int_0^a \mu_m(r)dr}$ and $\pi_2(a) = e^{-\int_0^a \mu_f(r)dr}$.

Proof of Lemma 4.1

For examples of integration on the characteristic lines, see [2] and [5]. We explain here, the details of the calculations.

Indeed, the equation (4.2) can be rewritten as

$$\begin{cases} -\partial_t l - \partial_a l + \mu_f l = V(a,t) & \text{in } Q, \\ l(a,T) = l_T(a) & \text{in } Q_A, \\ l(A,t) = 0 & \text{in } Q_T, \end{cases} \tag{4.5}$$

where $V(a,t) = (1-\gamma)\beta(a,p)n(0,t) + \gamma\beta(a,p)l(0,t)$ and n satisfies

$$\begin{cases} -\partial_t n - \partial_a n + \mu_m n = 0 & \text{in } Q, \\ n(a,T) = n_T(a), & \\ n(A,t) = 0 & \text{in } Q_T. \end{cases} \tag{4.6}$$

For $t \leqslant a$, set $w(s) = l(s, T+t_0+s)$ with $t = T+t_0+s$ and $s \in (0,A)$.
Thus

$$w'(s) = \partial_t l(s, T+t_0+s) + \partial_a l(s, T+t_0+s)$$

.

we obtain the following system:

$$\begin{cases} w'(s) = \mu_f(s)w(s) - V(s, T+t_0+s) & \text{in } Q, \\ w(-t_0) = l(-t_0; T) & \text{in } Q_A. \end{cases} \tag{4.7}$$

Using Duhamel's formula, the solution of (4.5 can be written as follows

$$w(s) = Ce^{\int_{-t_0}^{s} \mu_f(\tau)d\tau} - \int_{-t_0}^{s} e^{\int_\tau^s \mu_f(\theta)d\theta} V(\tau, T+t_0+\tau)d\tau$$

Taking into account the initial condition, we obtain $C = l(-t_0; T) = l(T-t+s, T)$. Thus

$$w(s) = \frac{\pi_2(T-t+s)}{\pi_2(s)} l(T-t+s, T) - \int_{T-t+s}^{s} \frac{\pi_2(\tau)}{\pi_2(s)} V(\tau, t-s+\tau)d\tau, \tag{4.8}$$

where $\pi_2(a) = e^{-\int_0^a \mu_f(r)dr}$.
Changing the variable $l = t-s+\tau$ and taking $s = a$ in (4.8), we obtain

$$l(a,t) = w(a) = \frac{\pi_2(T-t+a)}{\pi_2(a)} l(T-t+a, T) + \int_t^T \frac{\pi_2(a-t+s)}{\pi_2(a)} V(a-t+s, s)ds,$$

with $T-t \leqslant A-a$.

For $t > a$, set $w(s) = l(A+a_0+s, s)$ with $a = A+a_0+s$ and $s \in (0,T)$, then w satisfies

$$\begin{cases} w'(s) = \mu_f(A+a_0+s)w(s) - V(A+a_0+s, s) & \text{in } Q, \\ w(-a_0) = l(A; -a_0) & \text{in } (0,T). \end{cases} \tag{4.9}$$

Applying Duhamel's formula to the system (4.9), we have

$$w(s) = Ce^{\int_{-a_0}^{s} \mu_f(A+a_0+\tau)d\tau} - \int_{-a_0}^{s} e^{\int_\tau^s \mu_f(A+a_0+\theta)d\theta} V(A+a_0+\tau, \tau)d\tau \tag{4.10}$$

Since $l(A,t) = 0$ then $C = 0$ and (4.10) becomes

$$w(s) = -\int_{A-a+s}^{s} e^{\int_\tau^s \mu_f(a-s+\theta)d\theta} V(a-s+\tau, \tau)d\tau. \tag{4.11}$$

Making the change of variable $l = a - s + \theta$ and taking $s = t$ in (4.11), we obtain

$$l(a,t) = w(t) = \int_t^{A-a+t} \frac{\pi_2(a-s+t)}{\pi_2(a)} V(a-t+s,s)ds,$$

with $T - t > A - a$.
Thus we obtain (4.4).

The same procedure applied to the system (4.6) leads to (4.3). □

The system (4.1) is null approximately controllable. Indeed we have the following result:

Theorem 4.1. *Let us assume that assumptions* $(H_1)-(H_2)$ *hold. For every time* $T > \max\{a_1, b_1\} + \max\{A - a_2, A - b_2\}$*, for every* $\kappa, \nu > 0$ *and for every* $(m_0, f_0) \in \left(L^2(Q_A)\right)^2$*, there exists a control* (v_κ, v_ν) *such that the solutions* m *and* f *of the system* (4.1) *verify*

$$||m(.,T)||_{L^2(0,A)} \leq \kappa \quad \textit{and} \quad ||f(.,T)||_{L^2(0,A)} \leq \nu. \tag{4.12}$$

The main idea in this part is to establish an observability inequality of (4.2) that will allow us to prove the approximate null controllability of (4.1). The basic idea for establishing this inequality is the estimation of non-local terms.
For that, suppose that the assumptions (H_1), (H_2), (H_3) and (H_4) are fullfiled, then we have the following result.

Theorem 4.2. *Under the assumptions of Theorem* 2.1*, there exists a constant* $C_T > 0$ *nondepending on* p *such that the couple* (n, l) *solution of* (4.2) *verifies the following inequality:*

$$\int_0^A n^2(a,0)da + \int_0^A l^2(a,0)da \leq C_T \left(\int_{\Theta_1} n^2(a,t)dadt + \int_{\Theta_2} l^2(a,t)dadt \right) \tag{4.13}$$

for every $T > \max\{a_1, b_1\} + \max\{A - a_2, A - b_2\}$.

For the proof of Theorem 4.2, we state the following estimations of the non-local terms.

Proposition 4.1. *Under the assumptions of Theorem* 2.1*, there exists* $C > 0$ *such that*

$$\int_0^{T-\eta} n^2(0,t)dt \leq C \int_0^T \int_{a_1}^{a_2} n^2(a,t)dadt, \tag{4.14}$$

where $a_1 < \eta < T$.
In particular, for every $\varrho > 0$*, if* $a_1 = 0$ *and* $n_T(a) = 0$ *a.e* $a \in (0, \varrho)$*; there is* $C_{\varrho,T} > 0$ *such that:*

$$\int_0^T n^2(0,t)dt \leq C_\varrho \int_0^T \int_0^{a_2} n^2(a,t)dadt. \tag{4.15}$$

Moreover, if the first condition of (H_3) *holds, we have the inequality*

$$\int_0^{T-\eta} l^2(0,t)dt \leq C \int_{\Theta_2} l^2(a,t)dadt, \tag{4.16}$$

for every η *such that* $b_1 < b$ *and* $b_1 < \eta < T$.

Remark 4.1. *The first condition of the assumption* (H_3) *is not necessary for the proof of inequality* (4.14).

Proof of Proposition 4.1: See [14] and [11] □

Proposition 4.2. *Let us assume the assumptions* $(H_1)-(H_3)$. *For every* $T > \sup\{a_1, A-a_2\}$ *there exists* $C_T > 0$ *such that the solution* (n,l) *of the system* (4.1) *verifies the following observability inequality:*

$$\int_0^A n^2(a,0)da \leq C_T \int_{\Theta_1} n^2(a,t)dadt. \tag{4.17}$$

Note that for every $T > \sup\{a_1, A-a_2\}$, there exists $a_0 \in (a_1, a_2)$ such that $n(a,0) = 0$ for all $a \in (a_0, A)$. This is a consequence of the following lemma.

Lemma 4.2. *Let us suppose that* $T > \sup\{a_1, A-a_2\}$. *Then there exists* $a_0 \in (a_1, a_2)$ *such that* $T > A - a_0 > A - a$ *for all* $a \in (a_0, A)$. *Therefore,* $n(a,0) = 0$ *for all* $a \in (a_0, A)$.

Proof of Lemma 4.2
Suppose that $T > A-a_2$, then there exists $\kappa > 0$ (we choose κ such that $\kappa < a_2-a_1$) $T > A-a_2+\kappa$. So $T > A-(a_2-\kappa)$ and we denote $a_0 = a_2-\kappa$. Then, $T > A-a_0 > A-a$ for all $a \in (a_0, A)$. Finally, from (4.3) for all (a,t) such that $T-t > A-a$, we get $n(a,0) = 0$ for all $a \in (a_0, A)$. □

We also need the following estimate for the proof of the Theorem 4.2.

Proposition 4.3. *Let us assume the assumptions* $(H_1)-(H_2)$, *let* $b_1 < a_0 < b$ *and* $T > b_1$.
Then, there exists $C_T > 0$ *such that the solution* l *of* (4.2) *verifies the following observability inequality:*

$$\int_0^{a_0} l^2(a,0)da \leq C_T \int_{\Theta_2} l^2(a,t)dadt. \tag{4.18}$$

For the proof of Theorem 4.2, let $l = u_1 + u_2$ where u_1 and u_2 verify

$$\begin{cases} -\partial_t u_1 - \partial_a u_1 + \mu_f(a)u_1 = 0 & \text{in } (0,A)\times(0,T-\eta), \\ u_1(A,t) = 0 & \text{in } (0,T-\eta) \\ u_1(a,T-\eta) = l_\eta & \text{in } (0,A). \end{cases} \tag{4.19}$$

and

$$\begin{cases} -\partial_t u_2 - \partial_a u_2 + \mu_f(a)u_2 = V(a,t) & \text{in } (0,A)\times(0,T-\eta), \\ u_2(A,t) = 0 & \text{in } (0,T-\eta) \\ u_2(a,T-\eta) = 0 & \text{in } (0,A). \end{cases} \tag{4.20}$$

where the couple (n,l) verifies (4.2) with $l_\eta = l(a,T-\eta)$ in Q_A and

$$V(a,t) = \beta(a,p)l(0,t) + \beta(a,p)n(0,t).$$

Using Duhamel's formula we can write

$$u_2(a,t) = \int_t^{T-\eta} \mathbb{T}_{t-f}V(a,f)df$$

where $\mathbb{T}$ is the semigroup generated by the operator $-\dfrac{\partial}{\partial a} + \mu_f(a)$.

Proof of Theorem 4.2

We split the term to be estimated as follows

$$\int_0^A l^2(a,0)da = \int_0^{\max\{a_1,b_1\}} l^2(a,0)da + \int_{\max\{a_1,b_1\}}^A l^2(a,0)da.$$

As, $\max\{a_1,b_1\} < b$, using the Proposition 4.3 we obtain the estimate

$$\int_0^{\max\{a_1,b_1\}} l^2(a,0)da \le C\int_0^T\int_{b_1}^{b_2} l^2 a,t)dadt. \tag{4.21}$$

We are now left with the estimation of

$$\int_{\max\{a_1,b_1\}}^A l^2(a,0)da.$$

But since $l = u_1 + u_2$, we must therefore estimate

$$\int_{\max\{a_1,b_1\}}^A u_1^2(a,0)da + \int_{\max\{a_1,b_1\}}^A u_2^2(a,0)da.$$

We have

$$\int_{\max\{a_1,b_1\}}^A u_2^2(a,0)da \le C_{\eta,T}(A-\max\{a_1,b_1\})\left(\int_0^{T-\eta} l^2(0,t)dt + \int_0^{T-\eta} n^2(0,t)dt\right).$$

And then, using the Proposition 4.1, with $\eta = \max\{a_1,b_1\} + \delta < T$, $\delta > 0$, we obtain

$$\int_{\max\{a_1,b_1\}}^A u_2^2(a,0)da \le C_{\eta,T}\left(\int_0^T\int_{b_1}^{b_2} l^2(a,t)dadt + \int_0^T\int_{a_1}^{a_2} n^2(a,t)dadt\right). \tag{4.22}$$

As,

$$T > \max\{a_1,b_1\} + \max\{A-a_2, A-b_2\},$$

we can choose $\delta > 0$ small enough (δ should also check, $\max\{a_1,b_1\} < \min\{a_2,b_2\} - \delta$) such that

$$T > \max\{a_1,b_1\} + \max\{A-a_2, A-b_2\} + 2\delta;$$

therefore

$$T - (\max\{a_1,b_1\} + \delta) > A - (\min\{a_2,b_2\} - \delta).$$

Moreover for

$$a \in (\min\{a_2, b_2\} - \delta, A)$$

we have

$$T - (\max\{a_1, b_1\} + \delta) > A - (\min\{a_2, b_2\} - \delta) > A - a.$$

Then, from the Lemma 4.2

$$u_1(a, T) = 0 \text{ a.e. } a \in (\min\{a_2, b_2\} - \delta), A).$$

Therefore

$$\int_{\max\{a_1,b_1\}}^{A} u_1^2(a, 0)da = \int_{\max\{a_1,b_1\}}^{\min\{a_2,b_2\}-\delta} u_1^2(a, 0)da.$$

As

$$T - (\max\{a_1, b_1\} + \delta) > A - (\min\{a_2, b_2\} - \delta)),$$

then using the Proposition 4.2, we obtain

$$\int_{\max\{a_1,b_1\}}^{A} u_1^2(a, 0)da \leq K \int_0^{T-\eta} \int_{b_1}^{b_2} u_1^2(a, t)dadt. \tag{4.23}$$

As

$$u_1 = l - u_2,$$

then

$$\begin{aligned} &\int_0^{T-\eta} \int_{b_1}^{b_2} u_1^2(a, t)dadt \\ &\leq 2 \left(\int_0^{T-\eta} \int_{b_1}^{b_2} u_2^2 dadt + \int_0^{T-\eta} \int_{b_1}^{b_2} l^2(a, t)dadt \right). \end{aligned} \tag{4.24}$$

Moreover, under the assumption of Theorem 2.1, the solution u_2 of the system (4.20) verifies the following estimate :

$$\begin{aligned} \int_0^{T-\eta} \int_{b_1}^{b_2} u_2^2(a, t)dadt &\leq \int_0^{T-\eta} \int_0^{A} u_2^2(a, t)dadt \\ &\leq C \left(\int_0^{T-\eta} l^2(0, t)dt + \int_0^{T-\eta} n^2(0, t)dt \right). \end{aligned} \tag{4.25}$$

where $C = e^{\frac{3}{2}(T-\eta)} \|\beta\|_\infty^2 A$. From the Proposition 4.1, we get

$$\int_{\max\{a_1,b_1\}}^{A} u_1^2(a, 0)da \leq C(T, \eta, \|\beta\|_\infty) \left(\int_0^{T} \int_{b_1}^{b_2} l^2(a, t)dadt + \int_0^{T} \int_{a_1}^{a_2} n^2(a, t)dadt \right). \tag{4.26}$$

Combining the inequalities (4.22) and (4.26), we obtain

$$\int\limits_{\max\{a_1,b_1\}}^{A} l^2(a,0)da \leq C(T,\eta,\|\beta\|_\infty)\left(\int\limits_0^T\int\limits_{a_1}^{a_2} n^2(a,t)dadt + \int\limits_0^T\int\limits_{b_1}^{b_2} n^2(a,t)dadt\right). \tag{4.27}$$

Therefore, (4.21) and (4.27) give

$$\int\limits_0^A l^2(a,0)da \leq K_T\left(\int\limits_0^T\int\limits_{a_1}^{a_2} n^2(a,t)dadt + \int\limits_0^T\int\limits_{b_1}^{b_2} n^2(a,t)dadt\right). \tag{4.28}$$

Finally, combining (4.28) and the inequality of the Proposition 4.2, we get the observability inequality.

□

For $\epsilon > 0$ and $\theta > 0$, we consider the functional $J_{\epsilon,\theta}$ defined by:

$$J_{\epsilon,\theta}(v_m,v_f) = \frac{1}{2}\int_{\Theta_1} v_m^2 dadt + \frac{1}{2}\int_{\Theta_2} v_f^2 dadt + \frac{1}{2\epsilon}\int_0^A m^2(a,T)da + \frac{1}{2\theta}\int_0^A f^2(a,T)da, \tag{4.29}$$

where (m,f) is the solution of the following system (4.1).

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = \chi_{\Theta_1} v_m \quad \text{in } Q, \\ \partial_t f + \partial_a f + \mu_f f = \chi_{\Theta_2} v_f \quad \text{in } Q, \\ m(a,0) = m_0(a), \quad f(a,0) = f_0(a) \quad \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,p)f(a,t)da, \quad f(0,t) = \gamma\displaystyle\int_0^A \beta(a,p)f(a,t)da \quad \text{in } Q_T. \end{cases} \tag{4.30}$$

Lemma 4.3.

The functional $J_{\epsilon,\theta}$ is continuous, strictly convex and coercive. Consequently, $J_{\epsilon,\theta}$ reaches its minimum at a point $(v_{m,\epsilon}, v_{f,\theta}) \in L^2(\Theta_1)\times L^2(\Theta_2)$. Setting (m_ϵ, f_θ) the associated solution of (4.30) and (n_ϵ, l_θ) the solution of (4.2) with

$$n_\epsilon(a,T) = -\frac{1}{\epsilon}m_\epsilon(a,T) \quad \text{and} \quad l_\theta(a,T) = -\frac{1}{\theta}f_\theta(a,T),$$

we have

$$\chi_{\Theta_1} v_{m,\epsilon} = \chi_{\Theta_1} n_\epsilon \quad \text{and} \quad \chi_{\Theta_2} v_{f,\theta} = \chi_{\Theta_2} l_\theta.$$

Moreover, there exit $C_i > 0,\ 1 \leq i \leq 4$, independent of ϵ and θ such that

$$\int_{\Theta_1} n_\epsilon^2(a,t)dadt \leq C_1\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big),$$
$$\int_0^A m_\epsilon^2(a,T)da \leq \epsilon C_2\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big),$$
$$\int_{\Theta_2} l_\theta^2(a,t)dadt \leq C_3\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big),$$
$$\int_0^A f_\theta^2(a,T)da \leq \theta C_4\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big).$$

Proof of Lemma 4.3
It is easy to check that $J_{\epsilon,\theta}$ is coercive, continuous and strictly convex. Then, it admits a unique minimiser (v_ϵ, v_θ). The maximum principle gives

$$\chi_{\Theta_1} v_{m,\epsilon} = \chi_{\Theta_1} n_\epsilon \quad \text{and} \quad \chi_{\Theta_2} v_{f,\theta} = \chi_{\Theta_2} l_\theta \tag{4.31}$$

where the couple (n_ϵ, l_θ) is the solution of the system:

$$\begin{cases} -\partial_t n_\epsilon - \partial_a n_\epsilon + \mu_m n_\epsilon = 0 \quad \text{in } Q, \\ -\partial_t l_\theta - \partial_a l_\theta + \mu_f l_\theta = (1-\gamma)\beta(a,p) n_\epsilon(0,t) + \gamma\beta(a,p) l_\theta(0,t) \quad \text{in } Q, \\ n_\epsilon(a,T) = -\dfrac{1}{\epsilon} m_\epsilon(a,T), \quad l_\theta(a,T) = -\dfrac{1}{\theta} f_\theta(a,T) \quad \text{in } Q_A, \\ n_\epsilon(A,t) = 0, \quad l_\theta(A,t) = 0 \quad \text{in } Q_T. \end{cases} \tag{4.32}$$

Multiplying the first and the second equation of (4.32) by respectively m_ϵ and f_θ, integrating with respect to Q and using (4.31) we get

$$\int_{\Theta_1} n_\epsilon^2(a,t)dadt + \frac{1}{\epsilon}\int_0^A m_\epsilon^2(a,T)da = -\int_0^A m_0(a) n_\epsilon(a,0)da - (1-\gamma)\int_0^T\int_0^A \beta(a,p) f_\theta(a,t) n_\epsilon(0,t)dadt \tag{4.33}$$

and

$$\int_{\Theta_2} l_\theta^2(a,t)dadt + \frac{1}{\theta}\int_0^A l_\theta^2(a,T)da = -\int_0^A f_0(a) l_\theta(a,0)da + (1-\gamma)\int_0^T\int_0^A \beta(a,p) f_\theta(a,t) n_\epsilon(0,t)dadt. \tag{4.34}$$

Combining (4.33) and (4.34), we obtain

$$\begin{aligned} \int_{\Theta_1} n_\epsilon^2(a,t)dadt + \frac{1}{\epsilon}\int_0^A m_\epsilon^2(a,T)da + \int_{\Theta_2} l_\theta^2(a,t)dadt + \frac{1}{\theta}\int_0^A l_\theta^2(a,T)da = -\int_0^A m_0(a) n_\epsilon(a,0)da \\ - \int_0^A f_0(a) l_\theta(a,0)da. \end{aligned}$$

Using the Young's inequality, we have for any $\delta > 0$,

$$\begin{aligned} \int_{\Theta_1} n_\epsilon^2(a,t)dadt + \frac{1}{\epsilon}\int_0^A m_\epsilon^2(a,T)da + \int_{\Theta_2} l_\theta^2(a,t)dadt + \frac{1}{\theta}\int_0^A l_\theta^2(a,T)da \leq \frac{\delta}{2}\int_0^A m_0^2(a)da \\ + \frac{1}{2\delta}\int_0^A n_\epsilon^2(a,0)da + \frac{\delta}{2}\int_0^A f_0^2(a)da + \frac{1}{2\delta}\int_0^A l_\theta^2(a,0)da. \end{aligned}$$

Using the observability inequality (4.13) and choosing $\delta = C_T$ in the previous inequality, it follows that

$$\begin{aligned} \frac{1}{2}\int_{\Theta_1} n_\epsilon^2(a,t)dadt + \frac{1}{\epsilon}\int_0^A m_\epsilon^2(a,T)da + \frac{1}{2}\int_{\Theta_2} l_\theta^2(a,t)dadt + \frac{1}{\theta}\int_0^A l_\theta^2(a,T)da \leq \frac{C_T}{2}\Big(\int_0^A m_0^2(a)da \\ + \int_0^A f_0^2(a)da\Big). \end{aligned}$$

moreover, by seting

$$\kappa = \epsilon C_2\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big),$$
$$\nu = \theta C_4\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big),$$

and $(v_\kappa, v_\nu) = (v_{m,\epsilon}, v_{f,\theta})$.
which completes the proof of the theorem 4.1.
This gives the desired result necessary to the proof of the main one. □

Now, we consider the system

$$\begin{cases} \partial_t m_\epsilon(p) + \partial_a m_\epsilon(p) + \mu_m m_\epsilon(p) = \chi_{\Theta_1} n_\epsilon \quad \text{in } Q, \\ \partial_t f_\theta(p) + \partial_a f_\theta(p) + \mu_m f_\theta(p) = \chi_{\Theta_2} l_\theta \quad \text{in } Q, \\ m_\epsilon(p)(a,0) = m_0(a), \quad f_\theta(p)(a,0) = f_0(a) \quad \text{in } Q_A, \\ m_\epsilon(p)(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,p) f_\theta(p)(a,t)da, \quad f_\theta(p)(0,t) = \gamma \int_0^A \beta(a,p) f_\theta(p)(a,t)da \quad \text{in } Q_T, \end{cases} \tag{4.35}$$

where (n_ϵ, l_θ) is the solution of (4.32) that minimizes the functional $J_{\epsilon,\theta}$. We have the following result:

Lemma 4.4. *Under the assumptions of the Theorem 2.1, the solutions m_ϵ and f_θ verify the following inequalities:*

$$\int_0^A m_\epsilon^2(a,T)da + \int_0^T \int_0^A (1+\mu_m) m_\epsilon^2(a,t)dadt \le C\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big) \tag{4.36}$$

and

$$\int_0^A f_\theta^2(a,T)da + \int_0^T \int_0^A (1+\mu_f) f_\theta^2(a,t)dadt \le C\Big(\int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da\Big). \tag{4.37}$$

Proof of Lemma 4.4:
Let

$$y_\epsilon = e^{-\lambda_0 t} m_\epsilon \text{ and } z_\theta = e^{-\lambda_0 t} f_\theta,$$

where λ_0 is a positive constant that will be choose later.
The functions $y\epsilon$ and z_θ verify

$$\partial_t y_\epsilon + \partial_a y_\epsilon + (\lambda_0 + \mu_m) y_\epsilon = \chi_{\Theta_1} e^{-\lambda_0 t} n_\epsilon \tag{4.38}$$

and

$$\partial_t z_\theta + \partial_a z_\theta + (\lambda_0 + \mu_f) z_\theta = \chi_{\Theta_2} e^{-\lambda_0 t} l_\theta. \tag{4.39}$$

Multiplying the equality (4.38) and the equality (4.39) by respectively y_ϵ and z_θ and integrating with respect to Q, we get

$$\frac{1}{2}\int_0^A y_\epsilon^2(a,T)da + \frac{1}{2}\int_0^T y_\epsilon^2(A,t)dt + \int_0^T \int_0^A (\lambda_0 + \mu_m(a)) y_\epsilon^2(a,t)dadt = \frac{1}{2}\int_0^A y_0^2(a)da \tag{4.40}$$
$$+ (1-\gamma)^2 \int_0^T \left(\int_0^A \beta(a,p) z_\theta da\right)^2 dt + \int_0^T \int_0^A \chi_{\Theta_1} e^{-\lambda_0 t} n_\epsilon y_\epsilon dadt$$

and

$$\frac{1}{2}\int_0^A z_\theta^2(a,T)da + \frac{1}{2}\int_0^T z_\theta^2(A,t)dt + \int_0^T\int_0^A (\lambda_0+\mu_f(a))z_\theta^2(a,t)dadt = \frac{1}{2}\int_0^A f_0^2(a)da \quad (4.41)$$
$$+\gamma^2\int_0^T\Big(\int_0^A \beta(a,p)z_\theta da\Big)^2 dt + \int_0^T\int_0^A \chi_{\Theta_2}e^{-\lambda_0 t}l_\theta z_\theta dadt.$$

Using the Young's inequality, Cauchy Schwarz's inequality and the fact that β is L^∞, we prove that:

$$(1-\gamma)^2\int_0^T\Big(\int_0^A \beta(a,p)z_\theta da\Big)^2 dt + \int_0^T\int_0^A \chi_{\Theta_1}e^{-\lambda_0 t}n_\epsilon y_\epsilon dadt \le \alpha_+^2||z_\theta||^2_{L^2(Q)} + \frac{1}{2}||y_\epsilon||^2_{L^2(Q)} + \frac{1}{2}||n_\epsilon||^2_{L^2(\Theta_1)}$$

and

$$\gamma^2\int_0^T\Big(\int_0^A \beta(a,p)z_\theta da\Big)^2 dt + \int_0^T\int_0^A \chi_{\Theta_2}e^{-\lambda_0 t}l_\theta z_\theta dadt \le \alpha_+^2||z_\theta||^2_{L^2(Q)} + \frac{1}{2}||z_\theta||^2_{L^2(Q)} + \frac{1}{2}||l_\theta||^2_{L^2(\Theta_2)}.$$

Therefore, choosing $\lambda_0 > (\alpha_+^2 + 3/2)$, we get:

$$\frac{1}{2}\int_0^A z_\theta^2(a,T)da + \int_0^T\int_0^A (1+\mu_f(a))z_\theta^2(a,t)dadt \le \frac{1}{2}\Big(||f_0||^2_{Q_A} + ||l_\theta||^2_{L^2(\Theta_2)}\Big).$$

Finally, applying the result of Lemma 4.3 to the above inequality, it follows that

$$\frac{1}{2}\int_0^A z_\theta^2(a,T)da + \int_0^T\int_0^A (1+\mu_f(a))z_\theta^2(a,t)dadt \le C\Big(\int_0^A f_0^2(a)da + \int_0^A m_0^2(a)da\Big) \quad (4.42)$$

and then the inequality (4.37) holds.
Likewise, we have

$$\frac{1}{2}\int_0^A y_\epsilon^2(a,T)da + \int_0^T\int_0^A (1+\mu_m)y_\epsilon^2(a,t)dadt \le \frac{1}{2}||m_0||^2_{Q_A} + \alpha_+^2||z_\theta||^2_{L^2(Q)} + \frac{1}{2}||n_\epsilon||^2_{L^2(\Theta_1)}$$

Using the above inequality, Lemma 4.3 and the inequality (4.42) we obtain

$$\int_0^A y_\epsilon^2(a,T)da + \int_0^T\int_0^A (1+\mu_m)y_\epsilon^2(a,t)dadt \le C\Big(\int_0^A f_0^2(a)da + \int_0^A m_0^2(a)da\Big)$$

and then, we get the desired result.

□

Finally, from Lemma 4.3 and Lemma 4.4, if $(\epsilon,\theta)\longrightarrow(0,0)$ we get:

$$(\chi_{\Theta_1}n_\epsilon, \chi_{\Theta_2}l_\theta) \rightharpoonup (\chi_{\Theta_1}v_m, \chi_{\Theta_2}v_f) \text{ and } (m_\epsilon, f_\theta) \rightharpoonup (m,f),$$

with (m,f) solution of the problem (4.1) and

$$m(.,T) = f(.,T) = 0 \quad \text{a.e } a\in(0,A).$$

We have now the necessary ingredients for the proof of Theorem 2.1.

4.2 Proof of Theorem 2.1

In this section, we established the existence of a fixed point for the preceding auxiliary problem. Indeed, we consider that (H_3) hold and we suppose to simplify that $\lambda(0) = \lambda(A) = 0$. We define now the operator

$$\Lambda(p)\ : L^2(Q_T) \longrightarrow \mathcal{P}(L^2(Q_T)),\ p \longmapsto \Lambda(p)$$

where we designate the set $\Lambda(p)$ as above:

$$\Lambda(p) = \left\{ P(t) \in L^2(Q_T), \text{such that } P(t) = \int_0^A \lambda(a) m(p) da \right\}$$

where the couple $\big(m(p), f(p)\big)$ is the solution of the following system:

$$\begin{cases} \partial_t m(p) + \partial_a m(p) + \mu_m m(p) = \chi_{\Theta_1} n(p) \quad \text{in } Q, \\ \partial_t f(p) + \partial_a f(p) + \mu_m f(p) = \chi_{\Theta_2} l(p) \quad \text{in } Q, \\ m(p)(a,0) = m_0(a), \quad f(p)(a,0) = f_0(a) \quad \text{in } Q_A, \\ m(p)(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,p) f(p)(a,t) da, \quad f(p)(0,t) = \gamma \displaystyle\int_0^A \beta(a,p) f(p)(a,t) da \quad \text{in } Q_T, \end{cases} \tag{4.43}$$

and $\big(n(p), l(p)\big)$ the corresponding solution of the minimizer of $J_{\epsilon,\theta}$ with $m(p)(a,T) = f(p)(a,T) = 0$ for almost every $a \in (0, A)$.
It is obvious that $\Lambda(p)$ is convex.

Remark 4.2. *Note that since $m(p)$ depends on $f(p)$ through the system* (4.43) *then the set $\Lambda(p)$ also indirectly depends on $f(p)$.*

We have the following result.

Proposition 4.4. *Under the assumptions of the Theorem 2.1, for any $p \in L^2(Q_T)$ the solution of problem* (4.43) *satisfies*

$$|Y(t)| + \left|\left|\frac{d}{dt}Y\right|\right|_{L^2(0,T)} \le C\Big(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\Big),$$

where $Y(t) = \int_0^A \lambda(a)m(p)da$ and the constant C is independent of p, m_0 and f_0.

Proof of Proposition 4.4
Let $Y(t) = \displaystyle\int_0^A \lambda(a)m(p)da$. It is easy to prove that Y is solution of system

$$\begin{cases} \partial_t Y + \displaystyle\int_0^A \mu_m(a)\lambda(a)m(p)da = R(t) \quad \text{in } Q_T, \\ Y(0) = \displaystyle\int_0^A \lambda(a)m_0(a)da, \end{cases} \tag{4.44}$$

where

$$R(t) = \int_0^A \lambda'(a)m(p)da + (1-\gamma)\lambda(0)\int_0^A \beta(a,p)f(p)da + \int_0^{a_2} \lambda(a)n(p)da.$$

Using the Lemma 4.4 and the assumptions on β and λ, we infer that there exists $K>0$ such that

$$||R||_{L^2(Q_T)} \leq K\big(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\big). \tag{4.45}$$

By using (4.44), the Young's inequality and integrating on Q_T, we obtain

$$\int_0^T |\partial_t Y|^2 dt \leq 2\int_0^T |R(t)|^2 dt + 2\int_0^T \left(\int_0^A \mu_m(a)\lambda(a)m_\epsilon(p)da\right)^2 dt.$$

Moreover, the Cauchy Schwarz's inequality leads to

$$\int_0^T \left(\int_0^A \mu_m(a)\lambda(a)m_\epsilon(p)da\right)^2 dt \leq \int_0^A \mu_m(a)\lambda(a)da \int_0^T\int_0^A \mu_m(a)\lambda(a)m_\epsilon^2(p)dadt.$$

The inequality (4.36) and the fact that $\lambda \in C([0,A])$ give

$$\int_0^T\int_0^A \mu_m(a)\lambda(a)m_\epsilon^2(p)dadt \leq K_1\bigg(||m_0||^2_{L^2(Q_A)} + ||f_0||^2_{L^2(Q_A)}\bigg),$$

where $K_1>0$ is independent of p, ϵ and θ. Moreover as $\lambda\mu_m \in L^1(0,A)$, and using (4.45), it follows that

$$\left|\left|\frac{d}{dt}Y\right|\right|_{L^2(0,T)} \leq C\bigg(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\bigg). \tag{4.46}$$

Now, let $\tilde{Y} = e^{-\lambda_0 t}Y$. Then, $\tilde{Y}$ satisfies

$$\begin{cases} \partial_t\tilde{Y} + \lambda_0\tilde{Y} + e^{-\lambda_0 t}\displaystyle\int_0^A \mu_m(a)\lambda(a)m_\epsilon(p)da = e^{-\lambda_0 t}R(t) \quad \text{in } Q_T, \\ \tilde{Y}(0) = \displaystyle\int_0^A \lambda(a)m_0(a)da. \end{cases} \tag{4.47}$$

Multiplying the first equation of (4.47) by $\tilde{Y}$, integrating on $(0,t)$ and using successively Cauchy Schwarz and Young inequalities, we deduce that

$$|\tilde{Y}(t)|^2 + \lambda_0\int_0^t \tilde{Y}^2 dt \leq |\tilde{Y}(0)|^2 + \int_0^t \tilde{Y}^2 dt + \int_0^T \left(\int_0^A \mu_m(a)\lambda(a)m_\epsilon(p)da\right)^2 dt + ||R||_{L^2(Q_T)}.$$

Using the above calculations and choosing $\lambda_0 > 2$, we get

$$|\tilde{Y}(t)|^2 \leq K_2\bigg(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\bigg). \tag{4.48}$$

The desired result comes from (4.46) and (4.48). □

Let

$$W(0,T) = \left\{Y \in L^\infty(0,T),\ ||Y||_{L^\infty(0,T)} \leq \Upsilon;\ \left|\left|\frac{dY}{dt}\right|\right|_{L^2(0,T)} \leq \Upsilon\right\},$$

with $\Upsilon = C\Big(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)} \Big)$.

We have $W(0,T) \subset W^{1,1}(0,T)$. Moreover the injection of $W^{1,1}(0,T)$ into $L^2(0,T)$ is compact, see [4] Page 129. So $W(0,T)$ is relatively compact in $L^2(0,T)$. From Proposition 4.4 we have $\Lambda\big(W(0,T)\big) \subset W(0,T)$, and we see that $\Lambda\big(W(0,T)\big)$ is a relatively compact subset of $L^2(0,T)$.
Let us now prove that Λ is upper-semicontinuous. This is equivalent to prove that for any closed subset K of $L^2(0,T)$, $\Lambda^{-1}(K)$ is closed in $L^2(0,T)$. Let $(p_k) \in \Lambda^{-1}(K)$ such that p_k converges towards p in $L^2(0,T)$. Then, p_k is bounded and for all k there exists $P_k \in K$ such that $P_k \in \Lambda(p_k)$. Therefore, from the definition of Λ, there exists $(m_k, f_k) \in \big(L^2((0,T)\times(0,A))\big)^2$ associated to $(n_k, l_k) \in L^2(\Theta_1) \times L^2(\Theta_2)$ solution of (4.43) such that $P_k = \displaystyle\int_0^A \lambda(a) m_k(p_k)da$ and satisfying the inequalities of the Lemma 4.3 and Lemma 4.4. Consequently (m_k, f_k) and (n_k, l_k) are bounded respectively in $\big(L^2((0,T)\times(0,A))\big)^2$ and in $L^2(\Theta_1) \times L^2(\Theta_2)$. Thus, there exists a subsequences still denoted by (m_k, f_k) and (n_k, l_k) that converge weakly to (m, f) in $\big(L^2((0,T)\times(0,A))\big)^2$ and (n, l) in $L^2(\Theta_1) \times L^2(\Theta_2)$ respectively. Using hypothesis (H_3), it follows that $\displaystyle\int_0^A \lambda(a) m_k(p_k)da$ converges strongly to $\displaystyle\int_0^A \lambda(a) m(p)da$ in $L^2(0,T)$.
Now, by standard device we see that (m, f) associated to (n, l) are solution of (4.43) and satisfy the inequalities of the Lemma 4.3 and Lemma 4.4. This implies that $P \in \Lambda(p)$.
On the other hand, thanks to the Proposition 4.4, one can extract a subsequence also denoted by P_k that converges strongly towards the function P in $L^2(0,T)$. Since K is closed we deduce that $P \in K$. Finally, we deduce that $p \in \Lambda^{-1}(P)$.
Applying Kakutani's fixed point theorem [6] in the space $L^2(0,T)$ to the mapping Λ, we infer that there is at least one $Y \in W(0,T)$ such that $Y \in \Lambda(Y)$. This completes the null controllability proof of the model (2.1).

4.3 Proof of Theorem 2.2

4.3.1 Proof of Theorem 2.2-(1)

In this section, we always consider the following system:

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = \chi_\Theta v_m \quad \text{in } Q, \\ \partial_t f + \partial_a f + \mu_f f = 0 \quad \text{in } Q, \\ m(a,0) = m_0, \quad f(a,0) = f_0 \quad \text{in } Q_A, \\ m(0,t) = (1-\gamma)\int_0^A \beta(a,p) f da, \quad f(0,t) = \gamma \int_0^A \beta(a,p) f da \quad \text{in } Q_T, \end{cases} \tag{4.49}$$

for every p in $L^2(Q_T)$. Under the assumptions of Theorem 2.2, the controllability problem that is to find $v_m \in L^2(\Theta)$ such that (m, f) solution of the system (4.49) verifies

$$m(.,T) = 0 \quad a \in (\varrho, A)$$

is equivalent to the following observability inequality.

Proposition 4.5. *Let us assume the assumptions* $(H_1)-(H_2)-(H_3)$, *for every* $T > A - a_2$ *and for any* $\varrho > 0$, *if* $h(a,T) = h_T(a) = 0$ *a.e in* $(0,\varrho)$, *there exists* $C_{\varrho,T} > 0$ *such that the following inequality*

$$\int_0^A h^2(a,0)da + \int_0^A g^2(a,0)da \leq C_{\varrho,T} \int_{\Theta} h^2(a,t)dadt \tag{4.50}$$

holds, where (h,g) *is the solution of*

$$\begin{cases} -\partial_t h - \partial_a h + \mu_m h = 0 \quad in\, Q, \\ -\partial_t g - \partial_a g + \mu_f g = (1-\gamma)\beta(a,p)h(0,t) + \gamma\beta(a,p)g(0,t) \quad in\, Q, \\ h(a,T) = h_T, \quad g(a,T) = 0 \quad in\, Q_A, \\ h(A,t) = 0, \quad g(A,t) = 0 \quad in\, Q_T. \end{cases} \tag{4.51}$$

For the proof of the Proposition 4.5, we state the following estimate.

Proposition 4.6. *Under the assumptions* (H_1) *and* (H_2), *there exists a constant* $C > 0$ *such that the solution* (h,g) *of the system* (4.51) *verifies*

$$\int_0^A g^2(a,0)da + \int_0^T \int_0^A (1+\mu_f)g^2(a,t)dadt \leq C \int_0^T h^2(0,t)dt. \tag{4.52}$$

Moreover, we deduce for $h_T = 0$ *a.e in* $(0,\varrho)$ *that there exists a constant* $C_{\varrho,T} > 0$ *such that*

$$\int_0^A g^2(a,0)da \leq C_{\varrho,T} \int_0^T h^2(0,t)dt \leq C_{\varrho,T} \int_0^T \int_0^{a_2} h^2(a,t)dadt. \tag{4.53}$$

Proof of Proposition 4.5
Combining the inequality (4.17) of Proposition 4.2and the inequality (4.53 of Proposition 4.6, we have

$$\begin{aligned} \int_0^A h^2(a,0)da + \int_0^A g^2(a,0)da &\leqslant C_T \int_{\Theta_1} h^2(a,t)dadt + C_{\varrho,T} \int_0^T \int_0^{a_2} h^2(a,t)dadt \\ &\leqslant \max\Big\{C_T; C_{\varrho,T}\Big\}\Big(\int_{\Theta_1} h^2(a,t)dt + \int_0^T \int_0^{a_2} h^2(a,t)dadt\Big) \\ &\leqslant \max\Big\{C_T; C_{\varrho,T}\Big\}\Big(\int_{\Theta} h^2(a,t)dadt + \int_0^T \int_0^{a_2} h^2(a,t)dadt\Big) \\ &\leqslant C'_{\varrho,T} \int_{\Theta} h^2(a,t)dadt \end{aligned}$$

where we set $C'_{\varrho,T} = 2\max\Big\{C_T; C_{\varrho,T}\Big\}$.

□

Now, let $\epsilon > 0$ and $\varrho > 0$. We consider the functional J_ϵ defined by

$$J_\epsilon(v_m) = \frac{1}{2}\int_0^T \int_{a_1}^{a_2} v_m^2(a,t)dadt + \frac{1}{2\epsilon}\int_{\varrho}^A m^2(a,T)da, \tag{4.54}$$

where (m, f) is the solution of the following system

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = \chi_{\Theta_1} v_m & \text{in } Q, \\ \partial_t f + \partial_a f + \mu_f f = 0 & \text{in } Q, \\ m(a,0) = m_0(a) \quad f(a,0) = f_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma) \displaystyle\int_0^A \beta(a,p) f(a,t) da & \text{in } Q_T, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,p) f(a,t) da & \text{in } Q_T. \end{cases} \tag{4.55}$$

We have the following lemma.

Lemma 4.5. *The functional J_ϵ is continuous, strictly convex and coercive. Consequently, J_ϵ reaches its minimum at one has $v_{m,\epsilon} \in L^2(\Theta)$.*

Moreover, setting m_ϵ the associated solution of (4.55) and h_ϵ the solution of (4.51) with $h_\epsilon(a,T) = -\frac{1}{\epsilon}\chi_{\{(0,\varrho)\}} m_\epsilon(a,T)$ one has $v_{m,\epsilon} = \chi_\Theta h_\epsilon$ and there exists a positive constants C_1, C_2 independent of ϵ such that

$$\int_0^T \int_0^{a_2} h_\epsilon^2(a,t) da dt \leq C_1 \Big(\int_0^A m_0^2(a) da + \int_0^A f_0^2(a) da \Big)$$

and

$$\int_\varrho^A m_\epsilon^2(a,T) da \leq \epsilon C_2 \Big(\int_0^A m_0^2(a) da + \int_0^A f_0^2(a) da \Big).$$

Proof of Lemma 4.5
The proof is similar to that of Lemma 4.3. □

By making ϵ tending towards zero, we thus obtain that $\chi_\Theta h_\epsilon \rightharpoonup \chi_\Theta v_m$ and $(m\epsilon, f_\epsilon) \rightharpoonup (m, f)$, where (m, f) is the solution of the system (4.55) that verifies

$$m(.,T) = 0 \text{ a.e in } (\varrho, A).$$

Finally, a similar procedure as in the proof of Theorem 2.1 is followed to get the null controllability for the nonlinear problem.

4.3.2 Proof of Theorem 2.2-(2)

Let $p \in L^2(Q_T)$, under the assumptions of Theorem 2.2, the following controllability problem find $v_f \in L^2(\Theta)$ such that the solution of the system

$$\begin{cases} \partial_t f + \partial_a f + \mu_f f = \chi \Theta_2 v_f & \text{in } Q, \\ f(a,0) = f_0(a) & \text{in } Q_A, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,p) f(a,t) da & \text{in } Q_T \end{cases} \tag{4.56}$$

verifies

$$f(.,T) = 0 \text{ a.e in } (0, A).$$

is equivalent to the following observability inequality.

Proposition 4.7. *Let us assume true the assumptions* $(H_1)-(H_2)-(H_3)$*. For any* $T > a_1+A-a_2$ *there exists* $C_T > 0$ *such that*

$$\int_0^A g^2(a,0)da \leq C_T \int_{\Theta_2} g^2(a,t)dadt, \tag{4.57}$$

where g *is solution of the system*

$$\begin{cases} -\partial_t g - \partial_a g + \mu_f g = \gamma\beta(a,p)g(0,t) & in\ Q, \\ g(a,T) = g_T \quad in\ Q_A, \\ g(A,t) = 0 \quad in\ Q_T. \end{cases} \tag{4.58}$$

Proof of Proposition 4.7
Using the inequality (4.16) of Proposition 4.1, the result of Proposition 4.3 and the representation of the solution of the system (4.58), we get the desired result. □

To conclude, a similar procedure as in the proof of Theorem 2.1 leads to the null controllability result for the nonlinear problem. We omit all details because the extension is straightforward.
Consider now the following operator: $\Phi : L^2(Q_T) \to L^2(Q_T)$ definie by

$$p \longmapsto v_f(p) \longmapsto (p, f(v_f(p))) \longmapsto m(p, f(v_f(p), p)) \longmapsto \int_0^A \lambda(a)m(p, f(v_f(p), p))da,$$

where $(m(p, f(v_f(p))), f(p, v_f(p), p)))$ is the solution of the of the folloing system

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = 0 & \text{in } Q, \\ \partial_t f + \partial_a f + \mu_f f = \chi_{\Theta_2} v_f & \text{in } Q, \\ m(a,0) = m_0(a) \quad f(a,0) = f_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,p)f(a,t)da & \text{in } Q_T, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,p)f(a,t)da & \text{in } Q_T. \end{cases} \tag{4.59}$$

By appying Schauder's fixed point theorem, it follows that:

Lemma 4.6. *The operator* Φ *admits a fixed point.*

And therefore proves Theorem 2.2-(2)

After the total extinction of femal at time T we put $v_f \equiv 0$, the pair (m, f) is then the solution of the system.

$$\begin{cases} \partial_t m + \partial_a m + \mu_m m = 0 & \text{in } (0,A) \times (0,T+A), \\ \partial_t f + \partial_a f + \mu_f f = \chi_{\Theta_2 \times (0,T)} v_f & \text{in } (0,A) \times (0,T+A), \\ m(a,0) = m_0(a) \quad f(a,0) = f_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,p)f(a,t)da & \text{in } (0,T+A), \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,p)f(a,t)da & \text{in } (0,T+A). \end{cases} \tag{4.60}$$

with $M(t)=\int_0^A \lambda(a)m(a,t)da \quad a.e$ in $(0,A+T)$.

Integrating along the caracteristics lines, the solution m of (4.60) is given by:

$$m(a,t)=\begin{cases} \dfrac{\pi_1(a)}{\pi_1(a-t)}m_0(a-t) \quad \text{if} \quad t\leq a, \\ \\ \pi_1(a)\int_0^A \beta(a,M(t-a))f(a,t-a)da \quad \text{if} \quad a<t \end{cases} \tag{4.61}$$

where $(a,t)\in(0,A)\times(0,A+T)$. Moreover, we have $f(a,t)=0$ a.e for all $t>T$. Indeed, let $\widehat{f}=e^{-\lambda_0 t}f$ in (4.61), where λ_0 is a positive real wich, will be fixed later.

The function $\widehat{f}$ verifies

$$\widehat{f}_t+\widehat{f}_a+(\mu_f+\lambda_0)\widehat{f}=\chi_{\Theta_2\times(0,T)}v_f e^{-\lambda_0 t}.$$

integrating this equation over $(0,A)\times(T,\alpha)$ where $\alpha>T$ we obtain:

$$\frac{1}{2}\int_0^A \widehat{f}^2(a,\alpha)da+\frac{1}{2}\int_T^\alpha \widehat{f}^2(A,t)dt+\int_T^\alpha\int_0^A(\lambda_0+\mu_f(a))\widehat{f}^2(a,t)dadt=\frac{1}{2}\int_T^\alpha\left(\int_0^A\beta(a,M)\widehat{f}^2da\right)dt. \tag{4.62}$$

Using the assumption on β and choosing $\lambda_0=\frac{A\beta_0^2}{2}+1$, we aobtain $\widehat{f}(a,t)=0$ a.e in $(0,A)\times(T,\alpha)$. Then $f(a,t)=0$ a.e in $(0,A)\times(T,t)$. If $t\geqslant T+A$, we have $t-a\geqslant T+A-a>T$ for all $a\in(0,A)$, then $f(a,T+A-a)=0$ a.e in $(0,A)$ and then, one has

$$m(a,T+A)=\pi_1(a)\int_0^A\beta(a,M(T+A-a))f(a,T+A-a)da=0 \quad \text{a.e in} \quad (0,A).$$

References

[1] B. Ainseba, S. Anita, *Internal stabilizability for a reaction-diffusion problem modelling a predator-prey system,* Nonlinear analysis, 61 (2005), $491 - 501$

[2] B. Ainseba, *Age-Dependent Population Dynamics Diffusive Systems,*Mathematical Control and Related Fields, doi:10.3934/dcdsb.2004.4.1233

[3] S. Anita, *Analysis and Control of Age-Dependent Population Dynamics,* Mathematical Modeling: Theory and Applications, Springer-Science+Business Media, B.V.

[4] H. Brezis, *Analyse fonctionnelle : Théorie et Applications,* Masson, Paris, 1983

[5] I. Boutaayamou, G. Fragnelli, *A degenerate population system: Carleman estimates and controllability,* ScienceDirect, doi.org/10.1016/j.na.2019.111742

[6] K. Deimling, *Nonlinear Functional Analysis,* Springer Verlag, Berlin, 1985

[7] A. V Fursikov, O. Y Imanuvilov, *Controllability of evolution equations,* volume 34 of Lecture Notes Series. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996

[8] Y. He, B. Ainseba, *Exact null controllability of the Lobesia botrana model with diffusion,* J. Math. Anal. Appl. 409(2014) $530 - 543$

[9] M. Iannelli, J. Ripoll, *Two-sex age structured dynamics in a fixed sex-ratio population,* Nonlinear Analysis: Real World Applications 13 (2012) $2562 - 2577$

[10] D. Maity, *On the Null Controllability of the Lotka-Mckendrick System,* Mathematical Control and Related Fields, hal-01941486 AIMS, 2019

[11] Y. Simporé, O. Traoré, *Null controllability of a nonlinear age, space and two-sex population dynamics structured model.* Mathematical Control and Related Fields, doi: 10.3934/mcrf.2021052

[12] A. Traoré, B. Ainseba, O. Traoré, *On the existence of solution of a four-stage and age-structured population dynamics model,* Journal of Mathematics Analysis and Applications 495 (2021); 124699/doi.org/10.1016/j.jmaa.2020.124699

[13] A. Traoré, B. Ainseba, O. Traoré, *"Null Controllability of a Four Stage and Age-Structured Population Dynamics Model"*, Journal of Mathematics, vol. 2021, Article ID 5546150, 17 pages, 2021. https://doi.org/10.1155/2021/5546150

[14] A. Traoré, O. S. Sougué, Y. Simporé, O. Traoré *"Null Controllability of a Nonlinear Age Structured Model for a Two-Sex Population"*, Abstract and Applied Analysis, vol. 2021, Article ID 6666942, 20 pages, 2021. https://doi.org/10.1155/2021/6666942

[15] G. F. Webb, *Theory of nonlinear age-dependent population dynamics,* volume 89 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1985

[16] C. Zhao, M. Wang, P. Zhao, *Optimal control of harvesting for age-dependent predator-prey system,* Mathematical and Computer Modelling, 42 (2005), $573 - 584$