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Stabilization for 1D wave equation with delay term on the dynamical control

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Abstract : Using the frequency domain approach we prove the rational stability for a wave equation with delay term on the dynamical control, after establishing the strong stability and the lack of uniform stability.

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1 Introduction

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In this paper, we consider the following wave equation with delay term on the dynamical control

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = 0 \;\forall\; t \in (0,+\infty) \\ u_x(1,t) + \eta(t) = 0 \;\forall\; t \in (0,+\infty) \\ \eta_t(t) - u_t(1,t) + \beta_1\eta(t) + \beta_2\eta(t-\tau) = 0 \;\; \forall\; t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{R} \\ \eta(t-\tau) = f_0(t-\tau) \;\forall\; t \in (0,\tau), \end{cases} \tag{1.1}$$

where η stands for the dynamical control, $\tau > 0$ denotes the time delay, β_1 and β_2 are positive constants. Note that the initial data (u_0, u_1, f_0) belong to a suitable space. The damping of the system is made via the indirect damping mechanism.
It is well known that if $\beta_2 = 0$, that is to say, in the absence of delay, the energy of problem (1.1) decays polynomially to zero with the rate t^{-1}; see for instance Wehbe [10] for one dimensional case and Toufayli [9] for higher dimension. In this paper, staying on the one dimensional space, we purpose a dynamical boundary moment control η which contains a time delay term τ and we look for how to stabilize the system (1.1) using a frequency domain approach. To do that, we use properties of the stability of the undelayed one. To our knowledge polynomial stability with delay term has not yet been done, even if the initial system, that is, without the time delay, decays polynomially.

The paper is organized as follows: section 2 is devoted to the well posedness while the section 3 deals with the strong stability of problem (1.1); in section 4 we establish the non uniform stability, and finally in section 5 we prove the rational stability of problem (1.1).

Throughout this paper, we assume that

$$\beta_2 < \beta_1. \tag{1.2}$$

2 Well posedness

Here we study the well posedness for the problem (1.1) using the semigroup theory. In order to manage the parts of the problem (1.1) containing the delay term, in other words, to give to the term containing the delay a full notation, we set

$$z(\rho,t) = \eta(t-\tau\rho), \quad \rho \in (0,1),\; t > 0. \tag{2.1}$$

Let us set

$$s = t - \tau\rho. \tag{2.2}$$

On the one hand we can easily compute

$$\begin{aligned} z_t(\rho, t) &= \frac{\partial}{\partial t}\eta(s) \\ &= \frac{\partial s}{\partial t}\frac{\partial}{\partial s}\eta(s) \\ &= \eta'(s) \end{aligned}$$

which implies

$$z_t(\rho, t) = \eta'(t - \tau\rho). \tag{2.3}$$

On the other hand we have

$$\begin{aligned} z_\rho(\rho, t) &= \frac{\partial}{\partial \rho}\eta(s) \\ &= \frac{\partial s}{\partial \rho}\frac{\partial}{\partial s}\eta(s) \\ &= -\tau\eta'(s) \end{aligned}$$

which implies

$$z_t(\rho, t) = -\tau\eta'(t - \tau\rho). \tag{2.4}$$

Eliminating $\eta'(t - \tau\rho)$ in (2.3) and (2.4) leads to

$$\tau z_t(\rho, t) + z_\rho(\rho, t) = 0. \tag{2.5}$$

Otherwise from (2.1) one can write

$$z(1, t) = \eta(t - \tau). \tag{2.6}$$

From (2.6), the fourth equation of (1.1) may be rewritten as

$$\eta_t(t) - u_t(1, t) + \beta_1\eta(t) + \beta_2 z(1, t) = 0 \ \ \forall\ t \in (0, +\infty). \tag{2.7}$$

Moreover from (2.1) follows

$$z(0, t) = \eta(t) \tag{2.8}$$

and with the last equation of (1.1) we get

$$z(\rho, 0) = \eta(-\tau\rho) = f_0(-\tau\rho)$$

what means

$$z(\rho, 0) = f_0(-\tau\rho). \tag{2.9}$$

The problem (1.1) is now equivalently to

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 \text{ in } \ ]0,1[\times(0,+\infty) \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, \ \ \rho \in (0,1), \ \ t > 0 \\ u(0,t) = 0 \ \forall \ t \in (0,+\infty) \\ u_x(1,t) + \eta(t) = 0 \ \forall \ t \in (0,+\infty) \\ \eta_t(t) - u_t(1,t) + \beta_1\eta(t) + \beta_2 z(1,t) = 0 \ \ \forall \ t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \ \text{ in }]0,1[\ \text{ and } \ \eta(0) = \eta_0 \\ z(0,t) = \eta(t), \ \ t > 0 \\ z(\rho,0) = f_0(-\tau\rho), \ \ \rho \in (0,1). \end{cases} \tag{2.10}$$

The well posedness of problem (1.1) follows from standard semigroup theory.

If we denote by

$$\mathcal{U} = \Big(u, u_t, \eta, z\Big)^{\mathsf{T}},$$

one has from (2.10)

$$\mathcal{U}_t = (u_t, u_{tt}, \eta_t, z_t)^{\mathsf{T}} = \Big(u_t, u_{xx}, u_t(1,t) - \beta_1\eta(t) - \beta_2 z(1,t), -\tau^{-1}z_\rho\Big)^{\mathsf{T}}.$$

Therefore problem (2.10) can be rewritten as:

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U} \\ \mathcal{U}(0) = (u_0, u_1, \eta_0, f_0(-\cdot\tau))^{\mathsf{T}}, \end{cases} \tag{2.11}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\,(u,v,\eta,z)^{\mathsf{T}} = \left(v, u_{xx}, v(1) - \beta_1\eta - \beta_2 z(1), -\tau^{-1}z_\rho\right)^{\mathsf{T}},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \ (u,v,\eta,z)^{\mathsf{T}} \in \left(H^2(0,1)\cap V\right) \times V \times \mathbb{R} \times H^1(0,1) \ \middle| \ \begin{array}{l} z(0) = \eta \\ u_x(1) + \eta = 0 \end{array} \right\},$$

where

$$V = \left\{u \in H^1(0,1), u(0) = 0\right\}.$$

Denote by $\mathcal{H}$ the Hilbert space as below

$$\mathcal{H} = V \times L^2(0,1) \times \mathbb{R} \times L^2(0,1)$$

equipped with the norm

$$\left\|(u,v,\eta,z)^{\mathsf{T}}\right\|_{\mathcal{H}}^{2}=\|u_x\|_{L^2(0,1)}^2+\|v\|_{L^2(0,1)}^2+|\eta|^2+\zeta\|z\|_{L^2(0,1)}^2$$

where ζ is a positive constant verifying

$$\tau\beta_1<\zeta<\tau(2\beta_1-\beta_2) \tag{2.12}$$

and the natural associated inner product

$$\left\langle\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\z^*\end{pmatrix}\right\rangle_{\mathcal{H}}=\int_0^1\left(u_x\overline{u_x^*}+v\overline{v^*}\right)\,dx+\eta\overline{\eta^*}+\zeta\int_0^1 z(\rho)\overline{z^*(\rho)}\,d\rho.$$

We can now state the following existence results.

Theorem 2.1.
Assume that (1.2) *holds. Then for any datum* $U_0=(u_0,u_1,\eta_0,f_0)$ *belongs to* $\mathcal{H}$*, then the problem (1.1) has one and only one weak solution* $U=(u,u_t,\eta,z)$ *verifying:*

$$\begin{cases} u\in C\left([0,\infty),V\right)\cap C^1\left([0,\infty),L^2(0,1)\right)\\ \eta\in C\left([0,\infty)\right)\end{cases} \tag{2.13}$$

Moreover, if $U_0=(u_0,u_1,\eta_0,f_0)$ *belongs to* $\mathcal{D}(\mathcal{A})$*, then problem (1.1) has one and only one strong solution* $U=(u,u_t,\eta,z)$ *which satisfies*

$$\begin{cases} u\in C\left([0,\infty),H^2(0,1)\cap V\right)\cap C^1\left([0,\infty),V\right)\cap C^2\left([0,\infty),L^2(0,1)\right)\\ \eta\in C^1\left([0,\infty)\right).\end{cases} \tag{2.14}$$

Proof. We have

$$\begin{aligned}\left\langle\mathcal{A}\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix}\right\rangle_{\mathcal{H}} &= \left\langle\begin{pmatrix}v\\u_{xx}\\v(1)-\beta_1\eta-\beta_2 z(1)\\-\tau^{-1}z_\rho\end{pmatrix},\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix}\right\rangle_{\mathcal{H}}\\ &= \int_0^1 v_x\overline{u_x}\,dx+\int_0^1 u_{xx}\overline{v}\,dx+\left(v(1)-\beta_1\eta-\beta_2 z(1)\right)\overline{\eta}\\ &\quad -\zeta\tau^{-1}\int_0^1 z(\rho)\overline{z_\rho(\rho)}\,d\rho.\end{aligned}$$

Using Green formula, Cauchy Schwarz's inequality and the definition of $\mathcal{D}(\mathcal{A})$ we obtain

$$\begin{array}{rcl}\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}\right\rangle_{\mathcal{H}} & = & \Re\left(u_x(1,\cdot)\overline{v(1)} - u_x(0,\cdot)\overline{v(0)} + (v(1)-\beta_1\eta-\beta_2 z(1))\,\overline{\eta}\right) \\ & & -\frac{1}{2}\zeta\tau^{-1}\left|z(1)\right|^2 + \frac{1}{2}\zeta\tau^{-1}\left|z(0)\right|^2 \\ & = & -\Re\left(\eta\overline{v(1)}\right) + \Re\left((v(1)-\beta_1\eta-\beta_2 z(1))\,\overline{\eta}\right) - \frac{1}{2}\zeta\tau^{-1}\left|z(1)\right|^2 \\ & & +\frac{1}{2}\zeta\tau^{-1}\left|z(0)\right|^2 \\ & = & -\beta_1\left|\eta\right|^2 - \beta_2\left(z(1)\overline{\eta}\right) - \frac{1}{2}\zeta\tau^{-1}\left|z(1)\right|^2 + \frac{1}{2}\zeta\tau^{-1}\left|z(0)\right|^2 \\ & \leq & -\beta_1\left|\eta\right|^2 + \beta_2\left|\eta z(1)\right| - \frac{1}{2}\zeta\tau^{-1}\left|z(1)\right|^2 + \frac{1}{2}\zeta\tau^{-1}\left|z(0)\right|^2 \\ & \leq & -\beta_1\left|\eta\right|^2 + \frac{\beta_2}{2}\left|\eta\right|^2 + \frac{\beta_2}{2}\left|z(1)\right|^2 - \frac{\zeta}{2}\tau^{-1}\left|z(1)\right|^2 + \frac{\zeta}{2}\tau^{-1}\left|z(0)\right|^2.\end{array}$$

Recalling the definition of $\mathcal{D}(\mathcal{A})$ the above inequality becomes

$$\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}\right\rangle_{\mathcal{H}} \leq \left(-\beta_1 + \frac{\beta_2}{2} + \frac{\zeta}{2}\tau^{-1}\right)\left|\eta\right|^2 + \left(\frac{\beta_2}{2} - \frac{\zeta}{2}\tau^{-1}\right)\left|z(1)\right|^2.$$

Now the relation (2.12) allows to conclude that

$$\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}\right\rangle_{\mathcal{H}} \leq 0$$

which implies that the operator $\mathcal{A}$ is dissipative.
Let us prove that the operator $\lambda I - \mathcal{A}$ is surjective for at least one $\lambda > 0$.
For $(f, g, h, k)^{\mathsf{T}} \in \mathcal{H}$, we look for $(u, v, \eta, z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ solution of

$$\left\{\begin{array}{ll} \lambda u - v = f & \text{in }]0,1[\\ \lambda v - u_{xx} = g & \text{in }]0,1[\\ \lambda\eta - (v(1) - \beta_1\eta - \beta_2 z(1)) = h & \\ \lambda z + \tau^{-1} z_\rho = k & \text{in }]0,1[. \end{array}\right. \tag{2.15}$$

Assuming that we have found η and z with the appropriate regularity with the condition

$$z(0) = \eta.$$

We can determine z by solving the system

$$\left\{\begin{array}{ll} \tau^{-1} z_\rho + \lambda z = k & \text{in }]0,1[\\ z(0) = \eta. & \end{array}\right. \tag{2.16}$$

We obtain that

$$z(\rho)=\eta e^{-\lambda\tau\rho}+\tau e^{-\lambda\tau\rho}\int_0^{\rho}k(\sigma)e^{\lambda\tau\sigma}\,d\sigma. \tag{2.17}$$

and in particular

$$z(1)=\eta e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^{1}k(\sigma)e^{\lambda\tau\sigma}\,d\sigma.$$

The first equation of (2.15) directly involves

$$v=\lambda u-f. \tag{2.18}$$

Then using the third equation of (2.15), we find

$$\eta=\frac{\lambda u(1)}{\lambda+\beta_1+\beta_2e^{-\lambda\tau}}-\frac{f(1)+\beta_2\tau e^{-\lambda\tau}\int_0^1 k\left(\sigma\right)e^{\lambda\tau\sigma}\ d\sigma-h(1)}{\lambda+\beta_1+\beta_2e^{-\lambda\tau}}. \tag{2.19}$$

At this step it remains to determine u in order to conclude with proof. Indeed, as seen before, the determination of u involves that of the others components v, η and z. From the above, it follows that the function u verifies

$$\begin{cases} -u_{xx}+\lambda^2u=g+\lambda f & \text{in }]0,1[\\ u(0)=0 & \\ u_x(1)+\dfrac{\lambda u(1)}{\lambda+\beta_1+\beta_2e^{-\lambda\tau}}=\dfrac{f(1)+\beta_2\tau e^{-\lambda\tau}\int_0^1 k\left(\sigma\right)e^{\lambda\tau\sigma}\ d\sigma-h(1)}{\lambda+\beta_1+\beta_2e^{-\lambda\tau}} & \end{cases} \tag{2.20}$$

By using Lax-Milgram's Lemma, the problem (2.20) admits a unique weak solution. Indeed multiplying the first equation by $\phi\in V$ and by integrating formally by parts on $[0,1]$ we get

$$a(u,\phi)=F(\phi),\forall\ \phi\in V, \tag{2.21}$$

where the bilinear and continuous form a is given by

$$a(u,\phi)=\int_0^1\left(u_x\phi_x+\lambda^2u\phi\right)dx+\frac{\lambda u(1)}{\lambda+\beta_1+\beta_2e^{-\lambda\tau}}\phi(1)\quad\forall\ u,\phi\in V,$$

while the linear form F is

$$F(\phi)=\int_0^1(g+\lambda f)\phi\,dx+\frac{f(1)+\beta_2\tau e^{-\lambda\tau}\int_0^1 k\left(\sigma\right)e^{\lambda\tau\sigma}\ d\sigma-h(1)}{\lambda+\beta_1+\beta_2e^{-\lambda\tau}}\phi(1),\quad\forall\ \phi\in V.$$

Since a is clearly strongly coercive on V and F is continuous on V, by Lax-Milgram's Lemma, problem (2.20) admits a unique solution $u\in V$. By taking test functions $v\in\mathcal{D}(0;1)$, we recover the first identity of (2.20). This guarantees that u belongs to $H^2(0,1)$. Using now Green's formula, we see that u satisfies the third identity of (2.20).
Finally, we define v and η by setting

$$v=\lambda u-f\text{ and }\eta=-u_x(1)$$

This shows that the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and it generates a $\mathcal{C}_0$ semigroup of contractions in $\mathcal{H}$, under Lumer-Phillips theorem. So, we have found $(u,v,\eta,z)^T\in\mathcal{D}(\mathcal{A})$ which verifies (2.20). The proof ends by using the Hille-Yosida theorem. □

3 Strong stability

The main results of this section reads as follows.

Theorem 3.1.
The C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, that is for any $U_0 \in \mathcal{H}$,

$$\lim_{t\longrightarrow 0} \left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} = 0.$$

Proof. We use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [1, 3, 8]. Following this theory, since the resolvent of $\mathcal{A}$ is compact, it suffices to establish that $\mathcal{A}$ has no eigenvalue on the imaginary axis. For our purpose, it is easy to prove that the resolvent of the operator $\mathcal{A}$ defined in (2.11) is compact. We are ready now to achieve the proof of theorem 3.1 with the following lemma.

Lemma 3.2.
There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. By contradiction argument, we assume that there exists at least one $i\lambda \in \sigma(\mathcal{A})$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ on the imaginary axis. Let $U = (u, v, \eta, z)^{\mathsf{T}} \in D(\mathcal{A})$ be the corresponding normalized eigenvector, that is verifying $\|U\| = 1$ and

$$\mathcal{A}(u, v, \eta, z)^{\mathsf{T}} = i\lambda(u, v, \eta, z)^{\mathsf{T}}, \tag{3.1}$$

which is equivalent to

$$\begin{cases} v - i\lambda u = 0 \quad \text{in } (0,1) \\ u_{xx} - i\lambda v = 0 \quad \text{in } (0,1) \\ v(1,.) - \beta_1\eta - \beta_2 z(1,.) - i\lambda\eta = 0 \\ -\tau^{-1} z_\rho - i\lambda z = 0 \quad \text{in } (0,1). \end{cases} \tag{3.2}$$

Recalling the dissipativity of $\mathcal{A}$ and setting

$$\Lambda_1 = \beta_1 - \frac{\beta_2}{2} - \frac{\zeta\tau^{-1}}{2}, \quad \Lambda_2 = -\frac{\beta_2}{2} + \frac{\zeta\tau^{-1}}{2} \tag{3.3}$$

in the proof of theorem 2.1, it follows that

$$0 = \Re e\left\langle \mathcal{A}(u, v, \eta, z)^{\mathsf{T}}, (u, v, \eta, z)^{\mathsf{T}}\right\rangle_{\mathcal{H}} \quad \leq \quad -\Lambda_1 |\eta|^2 - \Lambda_2 |z(1)|^2 \leq 0 \ ; \tag{3.4}$$

that is

$$\begin{cases} z(1,.) = 0 \\ \eta = 0. \end{cases} \tag{3.5}$$

Owing to the definition of z in §2 we deduce that $\eta = z = 0$.

Now (3.2) becomes

$$\begin{cases} v - i\lambda u = 0 \quad \text{in } (0,1) \\ u_{xx} - i\lambda v = 0 \quad \text{in } (0,1) \\ v(1,.) = 0. \end{cases} \tag{3.6}$$

From the first equation of (3.6) we deduce that

$$u(1) = 0$$

Setting $v = i\lambda u$, it remains to find $u \in V$ which verifies

$$\begin{cases} u_{xx} + \lambda^2 u = 0 \quad \text{in } (0,1) \\ u_x(1) = 0 \\ u(1) = 0. \end{cases} \tag{3.7}$$

By Cauchy-Kowalevsky's theorem, there exists a nonempty neighbourhood $\mathcal{O}$ of 1 such that $u = 0$ in $\mathcal{O} \cap (0,1)$. Then the unicity theorem of Holmgren (see [4]) allows to conclude that

$$u = 0, \quad \text{on } (0,1). \tag{3.8}$$

We deduce that $(u, v, \eta, z)^\mathsf{T} = (0,0,0,0)^\mathsf{T}$ which contradicts the fact that $\|U\| = 1$. We conclude that $\mathcal{A}$ has no eigenvalue on the imaginary axis. □

As the conditions of the spectral decomposition theory of Sz-Nagy-Foias and Foguel are fully satisfied, the proof of theorem 3.1 is thus completed. □

4 Non uniform stability

In this section, we show that the semigroup generated by the operator $\mathcal{A}$ is not uniformly stable. For that we use the frequency domain approach (see Huang [5] and Prüss [7]), namely the below result.

Lemma 4.1.
A C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ of contractions on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable, i.e., satisfies

$$\left\|e^{t\mathcal{A}} U_0\right\|_{\mathcal{H}} \leq Ce^{-\omega t} \|U_0\|_{\mathcal{H}} \quad \forall\ U_0 \in \mathcal{H},\ \forall\ t \geq 0, \tag{4.1}$$

for some positive constants C and ω, if and only if

$$\rho(\mathcal{A}) \supset \{i\beta,\ \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{4.2}$$

and

$$\sup_{\beta \in \mathbb{R}} \left\|(i\beta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.3}$$

$\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

We state on the following result that constitutes the main of this section.

Theorem 4.2.
The system (1.1) is not exponentially stable in the energy space $\mathcal{H}$.

Proof. Following the lemma 4.1 above, we prove that the condition (4.3) is not satisfied in the sense that, there exists some sequences (λ_n), (U_n) and (F_n) such that

$$(i\lambda_n - \mathcal{A})U_n = F_n; \tag{4.4}$$

$$\|F_n\|_{\mathcal{H}} = O(1); \tag{4.5}$$

and

$$\lim_{n\to+\infty} \|U_n\|_{\mathcal{H}} = +\infty. \tag{4.6}$$

Let us set $U_n = (u^n, v^n, \eta^n, z^n)^{\mathsf{T}}$ and $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n})^{\mathsf{T}}$. The relation (4.4) is equivalent to

$$\begin{cases} i\lambda_n u^n - v^n = f_{1n} \\ i\lambda_n v^n - u^n_{xx} = f_{2n} \\ i\lambda_n \eta^n - (v^n(1) - \beta_1\eta^n - \beta_2 z^n(1)) = f_{3n} \\ i\lambda_n z^n + \tau^{-1} z^n_\rho = f_{4n}. \end{cases} \tag{4.7}$$

We look for a particular solution, defined for $f_{1n} = f_{3n} = f_{4n} = 0$, and f_{2n} will be chosen later. Then (4.7) becomes

$$\begin{cases} i\lambda_n u^n - v^n = 0 \\ i\lambda_n v^n - u^n_{xx} = f_{2n} \\ i\lambda_n \eta^n - (v^n(1) - \beta_1\eta^n - \beta_2 z^n(1)) = 0 \\ i\lambda_n z^n + \tau^{-1} z^n_\rho = 0. \end{cases} \tag{4.8}$$

The fourth equation of (4.8) combining with the condition $z(0) = \eta$ gives $z^n(\rho) = \eta e^{-i\lambda_n\tau\rho}$ that is

$$z^n(1) = \eta^n e^{-i\lambda_n\tau} \tag{4.9}$$

Eliminating v^n in the first and the second equation of (4.8), and using the fact that $(u^n, v^n, \eta^n, z^n)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$, it follows that

$$\begin{cases} u^n_{xx} + \lambda_n^2 u^n = -f_{2n} \\ u^n(0) = 0 \\ u^n_x(1) = -\eta. \end{cases} \tag{4.10}$$

The homogeneous equation associated to (4.10) can be solved as

$$u^n_h(x) = k_1\cos(\lambda_n x) + k_2\sin(\lambda_n x), \quad k_1, k_2 \in \mathbb{R}.$$

Notice that $u^n_1(x) = \cos(\lambda_n x)$ et $u^n_2(x) = \sin(\lambda_n x)$ are both the solutions of the homogeneous equation associated to (4.10). Let us denote by $\mathcal{W}(u^n_1, u^n_2)$ the "Wronskien" of the family (u^n_1, u^n_2). We have

$$\mathcal{W}(u^n_1, u^n_2) = \begin{vmatrix} \cos(\lambda_n x) & \sin(\lambda_n x) \\ -\lambda_n\sin(\lambda_n x) & \lambda_n\cos(\lambda_n x) \end{vmatrix} = \lambda_n \neq 0.$$

As $\mathcal{W}(u^n_1, u^n_2) \neq 0$, the family (u^n_1, u^n_2) forms a fundamental system of solutions. Consequently we can search the particular solution of (4.10) in the form

$$u^n_p(x) = k_1(x)\cos(\lambda_n x) + k_2(x)\sin(\lambda_n x) \tag{4.11}$$

where k_1 and k_2 are functions which verify

$$\begin{cases} k_1' \cos(\lambda_n x) + k_2' \sin(\lambda_n x) = 0 \\ -k_1' \lambda_n \sin(\lambda_n x) + k_2' \lambda_n \cos(\lambda_n x) = -f_{2n}. \end{cases} \tag{4.12}$$

The equation (4.12) can be solved as

$$k_1(x) = \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n s) ds \quad \text{and} \quad k_2(x) = -\frac{1}{\lambda_n} \int_0^x f_{2n}(s) \cos(\lambda_n s) ds. \tag{4.13}$$

Combining (4.13) and (4.11), we get

$$u_p^n(x) = -\frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin\left(\lambda_n (x - s)\right) ds. \tag{4.14}$$

Now the general solution of (4.10) can be written as

$$u^n(x) = k_1 \cos(\lambda_n x) + k_2 \sin(\lambda_n x) - \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin\left(\lambda_n (x - s)\right) ds, \quad k_1, k_2 \in \mathbb{R}. \tag{4.15}$$

On the one hand we have

$$u^n(0) = 0 \quad \Rightarrow \quad k_1 = 0.$$

On the other hand we compute

$$u^n(1) = k_2 \sin(\lambda_n) - \frac{1}{\lambda_n} \int_0^1 f_{2n}(s) \sin\left(\lambda_n (1 - s)\right) ds;$$

from which follows

$$k_2 = u^n(1) \frac{1}{\sin \lambda_n} + \frac{1}{\lambda_n \sin \lambda_n} \int_0^1 f_{2n}(s) \sin\left(\lambda_n (1 - s)\right) ds.$$

Consequently the general solution of (4.10) can be rewritten as

$$u^n(x) = u^n(1) \frac{\sin(\lambda_n x)}{\sin \lambda_n} + \frac{\sin(\lambda_n x)}{\lambda_n \sin \lambda_n} \int_0^1 f_{2n}(s) \sin\left(\lambda_n (1 - s)\right) ds - \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin\left(\lambda_n (x - s)\right) ds. \tag{4.16}$$

Differentiating the above relation it follows that

$$u_x^n(x) = \lambda_n u^n(1) \frac{\cos(\lambda_n x)}{\sin \lambda_n} + \frac{\cos(\lambda_n x)}{\sin \lambda_n} \int_0^1 f_{2n}(s) \sin\left(\lambda_n (1 - s)\right) ds - \int_0^x f_{2n}(s) \cos\left(\lambda_n (x - s)\right) ds$$

that is

$$u_x^n(1) = \lambda_n u^n(1) \cot \lambda_n + \cot \lambda_n \int_0^1 f_{2n}(s) \sin\left(\lambda_n (1 - s)\right) ds - \int_0^1 f_{2n}(s) \cos\left(\lambda_n (1 - s)\right) ds. \tag{4.17}$$

Now using (4.17) and the boundary condition $u_x(1) = -\eta$ we get

$$u^n(1) = -\frac{\eta^n \tan\lambda_n}{\lambda_n} - \frac{1}{\lambda_n}\int_0^1 f_{2n}(s)\sin(\lambda_n(1-s))\,ds + \frac{\tan\lambda_n}{\lambda_n}\int_0^1 f_{2n}(s)\cos(\lambda_n(1-s))\,ds. \quad (4.18)$$

From the first and the third equations of (4.8) we compute

$$\begin{aligned}\left(i\lambda_n + \beta_1 + \beta_2 e^{-i\lambda_n\tau}\right)\eta^n &= i\lambda_n u^n(1)\\ &= -i\eta^n \tan\lambda_n - i\int_0^1 f_{2n}(s)\sin(\lambda_n(1-s))\,ds\\ &\quad + i\tan\lambda_n \int_0^1 f_{2n}(s)\cos(\lambda_n(1-s))\,ds\end{aligned}$$

that is

$$\begin{aligned}\left(i\lambda_n + \beta_1 + \beta_2 e^{-i\lambda_n\tau} + i\tan\lambda_n\right)\eta^n &= -i\int_0^1 f_{2n}(s)\sin(\lambda_n(1-s))\,ds\\ &\quad + i\tan\lambda_n \int_0^1 f_{2n}(s)\cos(\lambda_n(1-s))\,ds.\end{aligned}$$

Let us set

$$\Pi = i\lambda_n + \beta_1 + \beta_2 e^{-i\lambda_n\tau} + i\tan\lambda_n.$$

Before computing η^n, let us demonstrate that $\Pi \neq 0$ with the choice $\lambda_n = 2n\pi + \frac{1}{\sqrt{n}}$.

We have

$$\begin{aligned}\Pi &= i\lambda_n + \beta_1 + \frac{\beta_2}{\cos(\lambda_n\tau) + i\sin(\lambda_n\tau)} + i\tan\lambda_n\\ &= i\lambda_n + \beta_1 + \frac{\beta_2}{\cos(\lambda_n\tau) + i\sin(\lambda_n\tau)} + i\tan\left(\frac{1}{\sqrt{n}}\right)\\ &= i\lambda_n + \beta_1 + \beta_2\left(\cos(\lambda_n\tau) - i\sin(\lambda_n\tau)\right) + i\tan\left(\frac{1}{\sqrt{n}}\right)\\ &= \beta_1 + \beta_2\cos(\lambda_n\tau) + i\left(\lambda_n - \sin(\lambda_n\tau) + \tan\left(\frac{1}{\sqrt{n}}\right)\right).\end{aligned}$$

So we can deduce that

$$\Pi = 0 \Longleftrightarrow \begin{cases} \beta_1 + \beta_2\cos(\lambda_n\tau) = 0\\ \lambda_n - \sin(\lambda_n\tau) + \tan\left(\frac{1}{\sqrt{n}}\right) = 0.\end{cases}$$

$$\begin{aligned}\beta_1 + \beta_2\cos(\lambda_n\tau) = 0 &\implies \cos(\lambda_n\tau) = -\frac{\beta_1}{\beta_2}\\ &\implies \cos(\lambda_n\tau) < -1 \text{ as } \beta_1 > \beta_2\end{aligned}$$

The last relation is impossible, so we have $\Pi \neq 0$.
We deduce that

$$\begin{aligned}\eta^n &= \frac{-i}{\Pi}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds \\ &+ \frac{i\tan\lambda_n}{\Pi}\int_0^1 f_{2n}(s)\cos\left(\lambda_n(1-s)\right)ds. \qquad (4.19)\end{aligned}$$

Inserting (4.19) in (4.18) it follows that

$$\begin{aligned}u^n(1) &= \frac{i\tan\lambda_n}{\lambda_n\Pi}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds - \frac{i\tan^2\lambda_n}{\lambda_n\Pi}\int_0^1 f_{2n}(s)\cos\left(\lambda_n(1-s)\right)ds \\ &-\frac{1}{\lambda_n}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds + \frac{\tan\lambda_n}{\lambda_n}\int_0^1 f_{2n}(s)\cos\left(\lambda_n(1-s)\right)ds\end{aligned}$$

in other words

$$\begin{aligned}u^n(1) &= -\frac{i\lambda_n+\beta_1+\beta_2 e^{-i\lambda_n\tau}}{\lambda_n\Pi}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds \\ &+\frac{\left(i\lambda_n+\beta_1+\beta_2 e^{-i\lambda_n\tau}\right)\tan\lambda_n}{\lambda_n\Pi}\int_0^1 f_{2n}(s)\cos\left(\lambda_n(1-s)\right)ds. \qquad (4.20)\end{aligned}$$

If we take λ_n large enough in (4.20) we get

$$u^n(1) \approx \frac{C_0}{\lambda_n}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds. \qquad (4.21)$$

Now let us compute $\lambda_n u^n(x)$, using (4.21). We get

$$\begin{aligned}\lambda_n u^n(x) &= \lambda_n u^n(1)\frac{\sin(\lambda_n x)}{\sin\lambda_n} + \frac{\sin(\lambda_n x)}{\sin\lambda_n}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds - \int_0^x f_{2n}(s)\sin\left(\lambda_n(x-s)\right)ds \\ &= C_0\frac{\sin(\lambda_n x)}{\sin\lambda_n}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds + \frac{\sin(\lambda_n x)}{\sin\lambda_n}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds \\ &\quad - \int_0^x f_{2n}(s)\sin\left(\lambda_n(x-s)\right)ds \\ &= (1-C_0)\frac{\sin(\lambda_n x)}{\sin\lambda_n}\int_0^1 f_{2n}(s)\sin\left(\lambda_n(1-s)\right)ds - \int_0^x f_{2n}(s)\sin\left(\lambda_n(x-s)\right)ds.\end{aligned}$$

Consequently we have

$$\lambda_n u^n(x) = \underbrace{(1-C_0)\frac{\sin(\lambda_n x)}{\sin\lambda_n}P(1)}_{K(x)}\underbrace{-P(x)}_{H(x)} \qquad (4.22)$$

where we set

$$P(x) := \int_0^x f_{2n}(s)\sin\left(\lambda_n(x-s)\right)ds. \qquad (4.23)$$

Let us choose $f_{2n}(x) := \sin(\lambda_n x)$. Then computing $P(x)$, we obtain

$$\begin{aligned}
P(x) &= \int_0^x \sin(\lambda_n s)\sin(\lambda_n(x-s))\,ds \\
&= \int_0^x \sin(\lambda_n s)\left(\sin(\lambda_n x)\cos(\lambda_n s) - \cos(\lambda_n x)\sin(\lambda_n s)\right) ds \\
&= \sin(\lambda_n x)\int_0^x \sin(\lambda_n s)\cos(\lambda_n s)ds - \cos(\lambda_n x)\int_0^x \sin^2(\lambda_n s)ds \\
&= \frac{\sin(\lambda_n x)}{2\lambda_n}\int_0^x \left(\sin^2(\lambda_n s)\right)' ds - \frac{\cos(\lambda_n x)}{2}\int_0^x (1-\cos(2\lambda_n s))\,ds \\
&= \frac{\sin^3(\lambda_n x)}{2\lambda_n} - \frac{x\cos(\lambda_n x)}{2} + \frac{\cos(\lambda_n x)\sin(2\lambda_n x)}{4\lambda_n} \\
&= \frac{\sin^3(\lambda_n x)}{2\lambda_n} - \frac{x\cos(\lambda_n x)}{2} + \frac{\cos^2(\lambda_n x)\sin(\lambda_n x)}{2\lambda_n} \\
&= \frac{\sin(\lambda_n x)}{2\lambda_n} - \frac{x\cos(\lambda_n x)}{2}.
\end{aligned}$$

Recalling the choice of λ_n, we have that $\sin(\lambda_n) \approx \frac{1}{\sqrt{n}}$, $\cos(\lambda_n) \approx 1$ and $\lambda_n \approx 2n\pi$. So we get

$$P(1) \approx \frac{1}{2\pi n^{3/2}} - \frac{1}{2} \approx -\frac{1}{2}.$$

Then it follow that

$$\begin{aligned}
\int_0^1 |H(x)|^2\,dx &\geq \int_0^1 \frac{x^2\cos^2(\lambda_n x)}{8}dx - \frac{C_1}{\lambda_n^2} \quad \text{(where } C_1 \text{ is a generic positive constant)} \\
&\geq \frac{1}{48} - \frac{C_1}{\lambda_n}.
\end{aligned}$$

In other words

$$\int_0^1 |H(x)|^2\,dx \geq \frac{1}{48} - \frac{C_1}{\lambda_n}. \tag{4.24}$$

Furthermore we have

$$\begin{aligned}
\int_0^1 |K(x)|^2\,dx &= \int_0^1 \left|(1-C_0)\frac{\sin(\lambda_n x)}{\sin\lambda_n}P(1)\right|^2 dx \\
&\geq \frac{C_2}{\sin^2\lambda_n}\int_0^1 \sin^2(\lambda_n x)dx \quad \text{(where } C_2 \text{ is a generic positive constant)} \\
&\geq C_2 n\left[\frac{x}{2} - \frac{\sin(2\lambda_n x)}{4\lambda_n}\right]_0^1
\end{aligned}$$

that is

$$\int_0^1 |K(x)|^2\,dx \geq C_3 n + C_4 \tag{4.25}$$

with C_3 (positive) and C_4 are generic constants.

Following (4.22) we have that

$$\begin{aligned} \int_0^1 |\lambda_n u^n(x)|^2\,dx &= \int_0^1 |K(x)+H(x)|^2 \\ &= \int_0^1 |K(x)|^2\,dx + \int_0^1 |H(x)|^2\,dx + 2\int_0^1 K(x)H(x)dx \end{aligned} \tag{4.26}$$

A straightforward calculation using the identity $2\alpha\beta \geq -\alpha^2 - \beta^2$ gives for all $\varepsilon > 0$:

$$\begin{aligned} KH &= \left(\frac{1}{\sqrt{\varepsilon}}K\right)\left(\sqrt{\varepsilon}H\right) \\ &\geq -\frac{K^2}{\varepsilon} - \varepsilon H^2. \end{aligned} \tag{4.27}$$

Inserting (4.27) in (4.26) it follows that

$$\int_0^1 |\lambda_n u^n(x)|^2 \geq \left(1-\frac{2}{\varepsilon}\right)\int_0^1 |K(x)|^2\,dx + (1-2\varepsilon)\int_0^1 |H(x)|^2\,dx. \tag{4.28}$$

Now combining (4.28), (4.25) and (4.24) we obtain

$$\int_0^1 |\lambda_n u^n(x)|^2 \geq C_3\left(1-\frac{2}{\varepsilon}\right)n + C_4\left(1-\frac{2}{\varepsilon}\right) + (1-\varepsilon)\left(\frac{1}{12}+\frac{C}{\lambda_n}\right).$$

Consequently there exists a positive constant γ_1, and another constant γ_2 such that

$$\int_0^1 |\lambda_n u^n(x)|^2\,dx \geq \gamma_1 n + \gamma_2. \tag{4.29}$$

We deduce from (4.29) that

$$\|U_n\|_{\mathcal{H}}^2 \geq \|v^n\|_{L^2(0,1)}^2 = \int_0^1 |\lambda_n u^n(x)|^2\,dx \geq \gamma_1 n + \gamma_2 \tag{4.30}$$

which implies

$$\lim_{n\to+\infty} \|U_n\|_{\mathcal{H}} = +\infty.$$

On the other hand, according the choice of F_n we have

$$\begin{aligned} \|F_n\|_{\mathcal{H}} &= \int_0^1 |f_{2n}(x)|^2\,dx \\ &= \int_0^1 \sin^2(\lambda_n x)dx \\ &= \frac{1}{2} - \frac{\sin(2\lambda_n)}{4\lambda_n} \end{aligned}$$

which implies

$$\|F_n\|_{\mathcal{H}} = O(1).$$

Finally we have found some sequences (λ_n), (U_n) and (F_n) which verify (4.4)-(4.6). Consequently system (1.1) is not uniformly stable. □

5 Rational stabilization result

Here we prove the problem (1.1) has a rational decays rate in the form t^{-1}. For that purpose we recall the following result due to Borichev and Tomilov [2]:

Lemma 5.1.
Let $\mathbf{A}$ *be the generator of a* C_0*-semigroup of bounded operators on a Hilbert space* $\mathbf{X}$ *such that* $i\mathbb{R} \subset \rho(\mathbf{A})$. *Then we have the polynomial decay*

$$\left\| e^{t\mathbf{A}} U_0 \right\| \leq \frac{C}{t^{1/\theta}} \left\| U_0 \right\|, \ t > 0,$$

if and only if

$$\limsup_{|\lambda| \to +\infty} \frac{1}{|\lambda|^\theta} \left\| (i\lambda - \mathbf{A})^{-1} \right\| < \infty.$$

The main result of this section is the next theorem.

Theorem 5.2.
The semigroup of system (1.1) *decays polynomially as*

$$\left\| e^{t\mathcal{A}} U_0 \right\| \leq \frac{C}{t} \left\| U_0 \right\|, \ \forall \ U_0 \in \mathcal{D}(\mathcal{A}), \ \forall \ t > 0. \tag{5.1}$$

Proof. It suffices to show following the results in [6, 10] and the above theorem, that for any $U = (u, v, \eta, z)^\top \in \mathcal{D}(\mathcal{A})$ and $F = (f, g, h, k)^\top \in \mathcal{H}$, the solution of

$$(i\lambda I - \mathcal{A}) U = F \tag{5.2}$$

verifies

$$\|U\|_{\mathcal{H}} \leq C\lambda \|F\|_{\mathcal{H}}; \tag{5.3}$$

where $\lambda > 0$ and C a positive constant.

Problem (1.1) without delay is the following one

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 \text{ in } \ ]0,1[\times(0,+\infty) \\ u(0,t) = 0 \ \forall \ t \in (0,+\infty) \\ u_x(1,t) + \eta(t) = 0 \ \forall \ t \in (0,+\infty) \\ \eta_t(t) - u_t(1,t) + \beta\eta(t) \ \ \forall \ t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \ \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{R} \end{cases}$$

which is well-posed in

$$\mathcal{H}_0 := V \times L^2(0,1) \times \mathbb{R} \tag{5.4}$$

endowed with the norm

$$\left\| (u, v, \eta)^\top \right\|_{\mathcal{H}_0}^2 := \|u_x\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 + \eta^2. \tag{5.5}$$

The generator of its semigroup is $\mathcal{A}_0$ defined by

$$\mathcal{A}_0 \left(u, v, \eta\right)^{\mathsf{T}} := \left(v,\ u_{xx},\ v(1) - \beta_1 \eta\right)^{\mathsf{T}} \tag{5.6}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) := \Big\{ \left(u, v, \eta\right)^{\mathsf{T}} \in \left(H^2(0,1) \cap V\right) \times V \times \mathbb{R}\ : u_x(1) + \eta = 0 \Big\}. \tag{5.7}$$

Thanks to [10], the operator $\mathcal{A}_0$ generates a polynomial stable semigroup with optimal decay rate t^{-1}. Therefore the solution $\left(u^*, v^*, \eta^*\right)^{\mathsf{T}}$ of

$$(i\lambda I - \mathcal{A}_0) \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \tag{5.8}$$

verifies

$$\left\| \left(u^*, v^*, \eta^*\right)^{\mathsf{T}} \right\|_{\mathcal{H}_0} \leq C\lambda \left\| \left(u, v, \eta\right)^{\mathsf{T}} \right\|_{\mathcal{H}_0} \tag{5.9}$$

where C is a positive constant.

On the other hand the system (5.8) can be rewritten as

$$\left\{ \begin{array}{l} i\lambda u^* - v^* = u \\ i\lambda v^* - u^*_{xx} = v \\ i\lambda \eta^* - v^*(1) + \beta_1 \eta^* = \eta. \end{array} \right. \tag{5.10}$$

With the help of integrations by parts and using (5.10) we have

$$\begin{aligned}
\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\0\end{pmatrix}\right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix}i\lambda u-v\\ i\lambda v-u_{xx}\\ i\lambda\eta-v(1)+\beta_1\eta+\beta_2 z(1)\\ i\lambda z+\tau^{-1}z_\rho\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\0\end{pmatrix}\right\rangle_{\mathcal{H}}\\
&= \int_0^1(\lambda u-v)_x\,\overline{u_x^*}dx+\int_0^1(i\lambda v-u_{xx})\,\overline{v^*}dx\\
&\quad+(i\lambda\eta-v(1)+\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\\
&= i\lambda\int_0^1 u_x\overline{u_x^*}dx-\int_0^1 v_x\overline{u_x^*}dx+i\lambda\int_0^1 v\overline{v^*}dx-\int_0^1 u_{xx}\overline{v^*}dx\\
&\quad+(i\lambda\eta-v(1)+\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\\
&= i\lambda\int_0^1 u_x\overline{u_x^*}dx-v(1)\overline{u_x^*(1)}+v(0)\overline{u_x^*(0)}+\int_0^1 v\overline{u_{xx}^*}dx\\
&\quad+i\lambda\int_0^1 v\overline{v^*}dx-u_x(1)\overline{v^*(1)}+u_x(0)\overline{v^*(0)}+\int_0^1 u_x\overline{v_x^*}dx\\
&\quad+(i\lambda\eta-v(1)+\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\\
&= i\lambda\int_0^1 u_x\overline{u_x^*}dx+v(1)\overline{\eta^*}+\int_0^1 v\overline{u_{xx}^*}dx+i\lambda\int_0^1 v\overline{v^*}dx\\
&\quad+\eta\overline{v^*(1)}+\int_0^1 u_x\overline{v_x^*}dx+(i\lambda\eta-v(1)+\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\\
&= \int_0^1 u_x\overline{(-i\lambda u^*+v^*)_x}dx+\int_0^1 v\overline{(-i\lambda v^*+u_{xx}^*)}dx\\
&\quad+\eta\overline{(-i\lambda\eta^*+v^*(1)-\beta_1\eta^*)}+(2\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\\
&= -\int_0^1|u_x|^2dx-\int_0^1|v|^2dx-|\eta|^2+(2\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\\
&= -\|u_x\|^2_{L^2(0,1)}-\|v\|^2_{L^2(0,1)}-|\eta|^2+(2\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}
\end{aligned}$$

Recalling (5.2) and using (5.5) we deduce from the above relation that

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0}=-\Re\left\langle F,\begin{pmatrix}u^*\\v^*\\\eta^*\\0\end{pmatrix}\right\rangle_{\mathcal{H}}+\Re\left((2\beta_1\eta+\beta_2 z(1))\,\overline{\eta^*}\right) \tag{5.11}$$

Applying Cauchy-Schwarz's and Young's inequalities, we get

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0}\le\|F\|_{\mathcal{H}}\left\|(u^*,v^*,\eta^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{4\beta_1^2}{\varepsilon}|\eta|^2+\frac{\beta_2^2}{\varepsilon}|z(1)|^2+\varepsilon|\eta^*|^2$$

As $\beta_2<\beta_1$ it follows that

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0}\le\|F\|_{\mathcal{H}}\left\|(u^*,v^*,\eta^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{4\beta_1^2}{\varepsilon}|\eta|^2+\frac{\beta_1^2}{\varepsilon}|z(1)|^2+\varepsilon|\eta^*|^2 \tag{5.12}$$

that is

$$\|U\|_{\mathcal{H}}^2 \leq \|F\|_{\mathcal{H}} \left\|(u^*, v^*, \eta^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0} + \frac{4\beta_1^2}{\varepsilon} |\eta|^2 + \frac{\beta_1^2}{\varepsilon} |z(1)|^2 + \varepsilon \left\|(u^*, v^*, \eta^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 + \zeta \|z\|_{L_2(0,1)}^2 . \quad (5.13)$$

Furthermore, following §2 (see the proof of theorem 2.1) we have

$$\Re \left\langle (i\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} = -\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq \Lambda_1 |\eta|^2 + \Lambda_2 |z(1)|^2 ;$$

where Λ_1 and Λ_2 are positive constants defined in (3.3). Consequently using the Cauchy-Schwarz inequality and the notation (5.2) it follows that

$$\Lambda_1 |\eta|^2 + \Lambda_2 |z(1)|^2 \leq \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}$$

that is

$$|\eta|^2 + |z(1)|^2 \leq (\min\{\Lambda_1, \Lambda_2\})^{-1} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} . \quad (5.14)$$

Now combining (5.13), (5.14) and (5.9) we get

$$\|U\|_{\mathcal{H}}^2 \leq C_1 \lambda \|F\|_{\mathcal{H}} \left\|(u, v, \eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0} + \frac{C_2}{\varepsilon} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + C_3 \lambda^2 \varepsilon \left\|(u, v, \eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 + \zeta \|z\|_{L_2(0,1)}^2 \quad (5.15)$$

where C_1, C_2 and C_3 are independent of λ and ε.
Furthermore we have

$$\left\|(u, v, \eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0} \leq \|U\|_{\mathcal{H}} . \quad (5.16)$$

Using (5.15), (5.16), and taking ε small enough (such that for example $C_3 \lambda^2 \varepsilon = o(1)$), it follows

$$\|U\|_{\mathcal{H}}^2 \leq C_1 \lambda \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \frac{C_2}{\varepsilon} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \zeta \|z\|_{L_2(0,1)}^2 . \quad (5.17)$$

Now we need a best estimation for $\zeta \|z\|_{L_2(0,1)}^2$.

Following (3.2) and solving the next Cauchy problem (5.18)

$$\begin{cases} \tau^{-1} z_\rho + i\lambda z = k \\ z(1) \end{cases} \quad (5.18)$$

we obtain

$$z(\rho) = z(1) e^{-i\lambda\tau(\rho-1)} - \tau \int_\rho^1 e^{-i\lambda\tau(\rho-\sigma)} k(\sigma)\, d\sigma, \quad \forall\, \rho \in (0,1) . \quad (5.19)$$

Using the triangular inequality, it follows from (5.19) that

$$|z(\rho)| \leq |z(1)| + \tau \int_\rho^1 |k(\sigma)|\, d\sigma, \quad \forall\, \rho \in (0,1) ,$$

that is

$$|z(\rho)|^2 \leq |z(1)|^2 + \tau^2 \left(\int_\rho^1 |k(\sigma)| \, d\sigma \right)^2 + 2 \, |z(1)| \, \tau \left(\int_\rho^1 |k(\sigma)| \, d\sigma \right), \quad \forall \, \rho \in (0,1) \,. \tag{5.20}$$

On the one hand, by Cauchy-Schwarz's inequality we obtain

$$\begin{aligned} \left(\int_\rho^1 |k(\sigma)| \, d\sigma \right)^2 & \leq \left(\int_\rho^1 |k(\sigma)|^2 \, d\sigma \right) \left(\int_\rho^1 d\sigma \right) \\ & \leq (1-\rho) \int_\rho^1 |k(\sigma)|^2 \, d\sigma \\ & \leq \int_\rho^1 |k(\sigma)|^2 \, d\sigma; \end{aligned}$$

that is

$$\left(\int_\rho^1 |k(\sigma)| \, d\sigma \right)^2 \leq \int_\rho^1 |k(\sigma)|^2 \, d\sigma. \tag{5.21}$$

On the other hand Young's inequality guarantees that

$$2 \, |z(1)| \, \tau \left(\int_\rho^1 |k(\sigma)| \, d\sigma \right) \leq |z(1)|^2 + \tau^2 \left(\int_\rho^1 |k(\sigma)| \, d\sigma \right)^2. \tag{5.22}$$

Combining (5.20), (5.21) and (5.22) it follows that

$$|z(\rho)|^2 \leq 2 \, |z(1)|^2 + 2\tau^2 \int_\rho^1 |k(\sigma)|^2 \, d\sigma. \tag{5.23}$$

Integrating (5.23) on $(0,1)$ and making easy computations we get

$$\zeta \, \|z\|^2_{L_2(0,1)} \leq 2\zeta \, |z(1)|^2 + 2\zeta\tau^2 \, \|k\|^2_{L_2(0,1)} \,.$$

By (5.14) we arrive at

$$\zeta \, \|z\|^2_{L_2(0,1)} \leq 2\zeta \left(\min\{\Lambda_1, \Lambda_2\} \right)^{-1} \|F\|_{\mathcal{H}} \, \|U\|_{\mathcal{H}} + 2\zeta\tau^2 \, \|F\|^2_{\mathcal{H}} \,. \tag{5.24}$$

Finally, combining (5.24) and (5.17) it follows that

$$\|U\|^2_{\mathcal{H}} \leq C\left(\lambda, \varepsilon\right) \left\|F\right\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + 2\zeta\tau^2 \|F\|^2_{\mathcal{H}} \tag{5.25}$$

where $C\left(\lambda, \varepsilon\right) = C_1\lambda + \dfrac{C_2}{\varepsilon} + 2\zeta \left(\min\{\Lambda_1, \Lambda_2\} \right)^{-1}$ with C_1 and C_2 independent of λ and ε.

Taking λ sufficiently large we get $\|U\|^2_{\mathcal{H}} \leq C \left(\lambda \, \|F\|_{\mathcal{H}} \, \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}} \right)$, from where follows that $\|U\|_{\mathcal{H}} \leq C\lambda \, \|F\|_{\mathcal{H}}$. Therefore recalling (5.2), we conclude that $\limsup\limits_{\lambda \to +\infty} \dfrac{1}{\lambda} \left\| (i\lambda - \mathcal{A})^{-1} \right\| < \infty$. So from Theorem 5.1 the semigroup decays polynomially with the rate t^{-1}. □

6 Conclusion

In this paper, we shown for $1D$ wave equation with delay term on the dynamical control, that the energy decays polynomially with the rate t^{-1}. To do this, we first prove the well posedness, the strong stability and the non uniform stability using the frequency domain approach.
For our future works, we intend to replace the delay (τ constant) by time-varying delay (i.e $\tau(t)$). We will also show that the energy of the system decays with the same rate t^{-1} assuming $\beta_1 > \beta_2$.

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