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# Existence and regularity of solutions in $\alpha$-norm for some nonlinear second order differential equation in Banach Spaces

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**Abstract :** Using the theory of cosine family, we prove the existence and regularity of solutions for some nonlinear second order differential equation in $\alpha$-norm. The delayed part is assumed to be locally lipschitz.

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## 1 Introduction

In this work, we study the existence and regularity of solutions for the following partial functional equation

$$\begin{cases} u''(t) = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0 \\ u_0 = \varphi \in \mathcal{C}_\alpha \\ u'_0 = \varphi' \in \mathcal{C}_\alpha, \end{cases} \tag{1.1}$$

where $A$ is the (possibly unbounded) infinitesimal generator of a strongly continuous cosine family of linear operators in X, $\mathcal{C}_\alpha = C^1([-r, 0], D((-A)^\alpha))$, $0 < \alpha \leq 1$, denotes the space of continuous functions from $[-r, 0]$ into $D((-A)^\alpha)$, $(-A)^\alpha$ is the fractional $\alpha$-power of A. This operator $((-A)^\alpha, D((-A)^\alpha))$ will be describe later. $\mathcal{C}_\alpha$is endowed with the following norm $\|h\|_{\mathcal{C}_\alpha} = |h|_\alpha + |h'|_\alpha$ for all $h \in \mathcal{C} = C^1([-r, 0], X)$, the norm $| \,.\, |_\alpha$ will be specified later. For every $t \geq 0$, $u_t$ denotes the history function of $\mathcal{C}$ defined by

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$$u_t(\theta) = u(t+\theta) \text{ for } \quad -r \leq \theta \leq 0,$$

$f : \mathbb{R}^+ \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ is a given function.
In [11] the authors study some semi-linear second order initial value problem. They also unify and simplify some ideas from the theory of strongly continuous cosine families of linear operators in Banach spaces. In [2], by using the theory of strongly continuous cosine families of linear operators in Banach, the author investigated the existence of solutions of the following semilinear second order differential initial value problem

$$\begin{cases} u''(t) = Au(t) + g(u(t), u'(t)) \text{ for } t \in [0,T] \\ \\ u(0) = u_0 \in X \quad u'(0) = u_1 \in X. \end{cases} \tag{1.2}$$

Using the theory of strongly continuous cosine families of linear operators in Banach space, we prove in this paper the existence of the mild and strict solutions. In [12], the author present a construction of cosine family with weak singularity in order to show its application to evolution Cauchy problems of the following form

$$\begin{cases} u''(t) = B^2 u(t) + g(t) \text{ for } t \in \mathbb{R} \\ \\ u(0) = u_0 \in X \quad u'(0) = u_1 \in X, \end{cases}$$

where X is a Banach space and $B : X \to X$ is a closed densely defined linear operator such that B and $-B$ generate analytic semigroups with weak singularity at 0.

The most fundamental work on cosine families can be found in [3, 4]. Important additions have also brought by Sova M. [8, 9] and Nagy B. [5, 6].

Our contribution in this topic is made in two steps. First, in [13], we generalize some results obtained in [11] by introducing a distributed delay on $[-r, 0],\ r > 0$. We also prove the existence and regularity of solutions of equation 1.1 in $\mathcal{C} = C^1([-r, 0], X)$. Secondly, the present work generalize the results obtained in [13] by using the fractional $\alpha$-power of A. Consequently, we obtain results which are more general than the ones obtained in [13]. Since the delay is distributed on $[-r, 0],\ r > 0$, this work generalize some results obtained in [10].

The organization of this work is as follows, in section 2, we collect some background materials required throughout the paper. In section 3, we study the existence of local mild solutions of equation (1.1) and we show the global continuation of solutions. We prove that in the case of local existence, the solutions blows up, we also show the continuous dependence with the initial data. In section 4, we will show the existence of strict solutions for equation (1.1). For illustration, we propose to study the existence of solutions for some partial functional equations with diffusion.

# 2 Preliminaries

Let $(X, \|.\|)$ be a Banach space and $\alpha$ be a constant such that $0 < \alpha < 1$ and $A$ be the infinitesimal generator of a strongly continuous cosine family of linear operators $(C(t))_{t \in \mathbb{R}}$ on X.

We assume without loss of generality that $0 \in \rho(-A)$. Note that if the assumption $0 \in \rho(-A)$ is not satisfied, one can substitute the operator $-A$ by the operator $(-A-\sigma)$ with $\sigma$ large enough such that $0 \in \rho(-A-\sigma)$. This allows us to define the fractional power $(-A)^\alpha$ for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D((-A)^\alpha)$ dense in X. The closeness of $(-A)^\alpha$ implies that $D((-A)^\alpha)$, endowed with the graph norm of $(-A)^\alpha$, $|x| = \|x\| + \|(-A)^\alpha x\|$, is a Banach space. Since $(-A)^\alpha$ is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|(-A)^\alpha x\|$. Thus, $D((-A)^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by $X_\alpha$.

**Definition 2.1.** *A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space $X$ into itself is called a strongly continuous cosine family if and only if*
*i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$,*
*ii) $C(0) = I$,*
*iii) $C(t)x$ is continuous in $t$ on $\mathbb{R}$ for each fixed $x \in X$.*

If $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine family in $X$, then $S(t)$ defined by

$$S(t)x = \int_0^t C(s)xds \text{ for } x \in X,\ t \in \mathbb{R}. \tag{2.1}$$

is a one parameter family of operators in $X$.

**Definition 2.2.** *The infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is the operator $A : X \to X$ defined by*

$$Ax = \frac{d^2C(t)x}{dt^2}|_{t=0}.$$

$D(A) = \{x \in X :$ *$C(t)x$ is a twice continuously differentiable function of* $t\}$.

We shall also make use of the set

$$E = \{x : \ C(t)x \text{ is a once continuously differentiable function of } t\}.$$

**Proposition 2.3.** *[11] Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family in $X$ with infinitesimal generator $A$. The following are true.*
*a) $D(A)$ is dense in $X$ and $A$ is a closed operator in $X$*
*b) if $x \in X$ and $r, s \in \mathbb{R}$, then $z = \int_r^s S(u)xdu \in D(A)$ and $Az = C(s)x - C(r)x$,*
*c) If $x \in X$ and $r, s \in \mathbb{R}$, then $z = \int_0^s \int_0^r C(u)C(v)xdudv \in D(A)$ and*
*$Az = \frac{1}{2}(C(r+s)x - C(s-r)x)$*
*d) if $x \in X$, then $S(t)x \in E$,*
*e) if $x \in X$, then $S(t)x \in D(A)$ and $\frac{dC(t)x}{dt} = AS(t)x$,*
*f) if $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{d^2C(t)x}{dt^2} = AC(t)x = C(t)Ax$,*
*g) if $x \in E$, then $\lim_{t\to 0} AS(t)x = 0$,*
*h) if $x \in E$, then $S(t)x \in D(A)$ and $\frac{d^2S(t)x}{dt^2} = AS(t)x$,*
*i) if $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$,*
*j) $C(t+s) - C(t-s) = 2AS(t)S(s)$ for all $s, t \in \mathbb{R}$.*

In [3], for $0 \leq \alpha \leq 1$, the fractional powers $(-A)^{\alpha}$ exist as closed linear operators in X,

$$D((-A)^{\beta}) \subset D((-A)^{\alpha}) \ \text{for} \ 0 \leq \alpha \leq \beta \leq 1, \ \text{and} \ (-A)^{\alpha}(-A)^{\beta} = (-A)^{\alpha+\beta}. \tag{2.2}$$

Throughout this paper, we assume that

($\mathbf{H_0}$) $A$ is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space $X$.

By Proposition 2.3, ($\mathbf{H_0}$) implies that the operator $A$ is densely defined in $X$, i.e. $\overline{D(A)} = X$. We have the following result.

**Proposition 2.4.** *[11] Assume that ($\boldsymbol{H_0}$) holds. Then there are constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|C(t)\| \leq Me^{-\omega|t|} \ \text{and} \ \|S(t) - S(t')\| \leq M\left|\int_{t'}^{t} e^{-\omega|s|}ds\right| \ \text{and for} \ t, t' \in \mathbb{R}.$$

From previous inequality, since $S(0) = 0$ we can deduce

$$\|S(t)\| \leq \frac{M}{\omega}e^{-\omega t} \text{ for } t \in \mathbb{R}^{+}.$$

In the sequel, let us pose $M_1 = \max\left(M, \dfrac{M}{\omega}\right)$.

**Theorem 2.5.** *[2] If $g : [0, T] \times X \times X \to X$ is continuous and $u$ is a solution of equation (1.2), then $u$ is a solution of the integral equation*

$$u(t) = C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)g(s, u(s), u'(s))ds \ \text{for} \ t \geq 0,$$

($\mathbf{A_1}$) For $0 < \alpha \leq 1$ $(-A)^{\alpha}$ maps onto X and is 1-1, so that $D((-A)^{\alpha})$ endowed with the norm the norm $|x|_{\alpha} = \|(-A)^{\alpha}x\|$ is a Banach space. We denote by $X_{\alpha}$ this Banach space. We further assume that $(-A)^{-1}$ is compact. We require the following lemmas.

**Lemma 2.6.** *[10] Assume ($\boldsymbol{H_0}$) holds. Then the following are true.*
*(i) For $0 < \alpha < 1$, $A^{-\alpha}$ is compact if only if $(-A)^{-1}$ is compact.*
*(ii) For $0 < \alpha < 1$ and $t \in \mathbb{R}$, $(-A)^{\alpha}C(t) = C(t)(-A)^{\alpha}$ and $(-A)^{\alpha}S(t) = S(t)(-A)^{\alpha}$.*

Recall from [3], $(-A)^{-\alpha}$ is given by the following formula

$$(-A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} t^{\alpha-1}(tI - A)^{-1}dt.$$

**Lemma 2.7.** *[10] Assume that ($\boldsymbol{H_0}$) holds, let $v : \mathbb{R} \to X$ such that $v$ is continuously differentiable and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then*
*(i) $q$ is twice continuously differentiable and for $t \in \mathbb{R}$,*

$$q(t) \in D(A), \;\; q'(t) = \int_0^t C(t-s)v(s)ds$$

*and*

$$q''(t) = \int_0^t C'(t-s)v(s)ds + C(0)v(t) = Aq(t) + v(t).$$

*(ii) For $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, $(-A)^{\alpha-1}q'(t) \in E$.*

**Theorem 2.8.** *(Heine's Theorem). Let $f$ be a continuous function on a compact set $K$, then $f$ is uniformly continuous on $K$.*

**Theorem 2.9.** *(Schauder's fixed point Theorem). Let $X$ be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f : K \to K$ there exists $x \in K$ such that $f(x) = x$.*

**Theorem 2.10.** *(Arzelà-Ascoli Theorem). Let $(X, d_X)$ and $(Y, d_Y)$ be compact metric spaces, $\mathcal{C}(X, Y)$ be the set of continuous functions from $X$ to $Y$ and let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$. If $\mathcal{F}$ is closed and equicontinuous then it is compact.*

# 3 Local existence, global continuation and blowing up of solutions

**Proposition 3.1.** *Assume that ($\boldsymbol{H_0}$) holds. If $u$ is a solution of equation (1.1), then*

$$u(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)f(s, u_s, u'_s)ds \text{ for } t \geq 0, \tag{3.1}$$

**Proof**. It is just a consequence of Theorem 2.5. In fact, let us pose $g(t, u(t), u'(t)) = f(t, u_t, u'_t)$ for $t \geq 0$. Then we get the desired result. ■

**Remark** : The converse is not true. In fact if $u$ satisfies equation (3.1), $u$ may be not twice continuously differentiable, that is why we distinguish between mild and strict solutions.

**Definition 3.2.** *We say that a continuous function $u : [-r, +\infty[ \to X_\alpha$ is a strict solution of equation (1.1) if the following conditions hold*
*(i) $u \in C^1([0, +\infty[; X_\alpha) \cap C^2((0, +\infty[; X_\alpha)$.*
*(ii) $u$ satisfies equation (1.1) on $[0, +\infty[$.*
*(iii) $u(\theta) = \varphi(\theta)$ for $-r \leq \theta \leq 0$.*

**Definition 3.3.** *We say that a continuous function* $u : [-r, +\infty[\to X_\alpha$ *is a mild solution of equation* (1.1) *if* $u$ *satisfies the following equation*

$$\begin{cases} u(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \displaystyle\int_0^t S(t-s)f(s, u_s, u'_s)ds \text{ for } t \geq 0 \\ u_0 = \varphi \in \mathcal{C}_\alpha. \\ u'_0 = \varphi' \in \mathcal{C}_\alpha. \end{cases}$$

In the following, we give a local existence of mild solutions of equation (1.1). First of all, we study the existence of mild solutions, in order to do that, we assume the following assumptions.

($\mathbf{H_1}$) The function $f : [0, b] \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ satisfies the following conditions
 i) $f : [0, b] \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ is continuously differentiable.
 ii) There exists a continuous nondecreasing function $\beta : [0, b] \to \mathbb{R}^+$ such that

$$\|f(t, \varphi, \varphi')\| \leq \beta(t)|\varphi|_\alpha \text{ for } (t, \varphi) \in [0, b] \times \mathcal{C}_\alpha.$$

($\mathbf{H_2}$) $A^{-1}$ is compact on $X$.

**Theorem 3.4.** *Assume that* ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$) *and* ($\boldsymbol{H_2}$) *hold. Let* $\varphi \in \mathcal{C}_\alpha$ *such that* $\varphi(0) \in D(A)$ *and* $\varphi'(0) \in E$ *and assume that*

$$\|(-A)^{\alpha-1}\| \sup_{t\in[0,b]} \left[\beta(t)(2Me^{-\omega b} + 1) + Me^{-\omega b}\right] < 1.$$

*Then equation* (1.1) *has at least one mild solution on* $[0, b]$.

**Proof**. Let $k > |\varphi|_{\mathcal{C}_\alpha}$, we define define the following set

$$Z_k = \{x \in C([0, b], X_\alpha) : \ x(0) = \varphi(0) \text{ and } |x|_\infty \leq k\},$$

where $|x|_\infty = \sup_{t\in[0,b]} |x(t)|_\alpha$. For $x \in Z_k$, define the $\widetilde{x}(t) : [0, b] \to X_\alpha$ by

$$\widetilde{x}(t) = \begin{cases} x(t) \text{ for } t \in [0, b] \\ \varphi(t) \text{ for } t \in [-r, 0] \end{cases}$$

The function $t \to \widetilde{x}_t$ is continuous from $[0, b]$ to $\mathcal{C}_\alpha$. Now, define the operator $\mathcal{H}$ on $Z_k$ by

$$\mathcal{H}(x)(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)f(s, x_s, x'_s)ds \text{ for } t \in [0, b].$$

It is sufficient to show that $\mathcal{H}$ has a fixed point in $Z_k$. We give the proof in several steps.

**Step 1 :** There is a positive $k > |\varphi|_{\mathcal{C}_\alpha}$ such that $\mathcal{H}(Z_k) \subset Z_k$.

If not, then for each $k > \|\varphi\|_{\mathcal{C}_\alpha}$, there exist $x_k \in Z_k$ and $t_k \in [0,b]$ such that $|(\mathcal{H}x_k)(t_k)|_\alpha > k$. Then by Proposition 2.3, we have

$$\begin{aligned}
k &< |(\mathcal{H}x_k)(t_k)|_\alpha \leq |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big|\int_0^{t_k} S(t_k-s)f(s,\widetilde{x}_s,\widetilde{x'}_s)ds\Big|_\alpha \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big\| -(-A)^{\alpha-1}\int_0^{t_k} AS(t_k-s)f(s,\widetilde{x}_s,\widetilde{x'}_s)ds\Big\| \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big\|(-A)^{\alpha-1}\Big[\int_0^{t_k}\frac{d}{ds}\Big(C(t_k-s)f(s,\widetilde{x}_s,\widetilde{x'}_s)\Big)ds - \int_0^{t_k} C(t_k-s)\frac{d}{ds}\Big(f(s,\widetilde{x}_s,\widetilde{x'}_s)\Big)ds\Big]\Big\| \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big\|(-A)^{\alpha-1}\Big(f(t_k,\widetilde{x}_{t_k},\widetilde{x'}_{t_k}) - C(t_k)f(0,\widetilde{x}_0,\widetilde{x'}_0)\Big)\Big\| \\
&\quad + \|(-A)^{\alpha-1}\|Me^{-\omega b}\|f(t_k,\widetilde{x}_{t_k},\widetilde{x'}_{t_k}) - f(0,\widetilde{x}_0,\widetilde{x'}_0)\| \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\|\Big(\|f(t_k,\widetilde{x}_{t_k},\widetilde{x'}_{t_k})\| + \|C(t_k)f(0,\widetilde{x}_0,\widetilde{x'}_0)\| + Me^{-\omega b}\|f(t_k,\widetilde{x}_{t_k},\widetilde{x'}_{t_k}) - f(0,\widetilde{x}_0,\widetilde{x'}_0)\|\Big) \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\|\Big([\beta(t_k) + Me^{-\omega b}]|\widetilde{x}_t|_{\mathcal{C}_\alpha} + 2Me^{-\omega b}\beta(0)|\widetilde{x}_t|_{\mathcal{C}_\alpha}\Big).
\end{aligned}$$

Since $|\widetilde{x}_s|_{\mathcal{C}_\alpha} \leq k$ for all $s \in [0,b]$ and $x \in Z_k$, then we obtain

$$k < M_1e^{-\omega b}\Big(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha\Big) + \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\beta(t)(2Me^{-\omega b}+1) + Me^{-\omega b}\Big]k.$$

Consequently

$$1 < \frac{M_1e^{-\omega b}\Big(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha\Big)}{k} + \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\beta(t)(2Me^{-\omega b}+1) + Me^{-\omega b}\Big].$$

It follows that when $k \to +\infty$ that

$$1 \leq \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\beta(t)(2Me^{-\omega b}+1) + Me^{-\omega b}\Big],$$

which gives a contradiction.

**Step 2 :** $\mathcal{H}$ is continuous

Let $(x^n)_n \in Z_k$ with $x^n \to x$ in $Z_k$. Then, the set

$$\Delta = \{(s,\widetilde{x}^n_s,\widetilde{x'}^n_s),\ (s,\widetilde{x}_s,\widetilde{x'}_s):\ s\in[0,b],\ n\geq 1\}$$

is compact in $[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha$. By Heine's theorem implies that $f$ is uniformly continuous in $\Delta$ and

$$\begin{aligned}
|\mathcal{H}(x^n)(t) - \mathcal{H}(x)(t)|_\infty &\leq \sup_{t\in[0,b]}\Big\| -(-A)^{\alpha-1}\int_0^t AS(t-s)(f(s,x^n_s,x'^n_s) - f(s,x_s,x'_s))ds\Big\| \\
&\leq \sup_{t\in[0,b]}\Big\|(-A)^{\alpha-1}\Big[\int_0^t\frac{d}{ds}\Big(C(t-s)(f(s,x^n_s,x'^n_s) - f(s,x_s,x'_s)\Big)ds \\
&\quad - \int_0^t C(t_k-s)\frac{d}{ds}\Big((f(s,x^n_s,x'^n_s) - f(s,x_s,x'_s)\Big)ds\Big]\Big\| \\
&\leq \|(-A)^{\alpha-1}\|\Big((1+M_1e^{-\omega b})\|f(t,x^n_t,x'^n_t) - f(t,x_t,x'_t)\| + 2M_1e^{-\omega b}\|f(0,x^n_0,x'^n_0) - f(0,x_0,x'_0)\|\Big) \to 0 \text{ as } n\to+\infty,
\end{aligned}$$

and this yield the continuity of $\mathcal{H}$ on $Z_k$.

**Step 3 :** The set $\{\mathcal{H}(x)(t) :\ x \in Z_k\}$ is relatively compact for each $t \in [0, b]$.

Let $t \in ]0, b]$ be fixed, using the same reasoning like in the **Step 1**, we have

$$\|(-A)^{\alpha}\mathcal{H}(x))(t)\| \quad \leq \quad \|(-A)^{\alpha-1}\|\Big[M_1 e^{-\omega b}\Big(\|A\varphi(0)\| + \|A\varphi'(0)\|\Big) + \sup_{t\in[0,b]}\Big(\beta(t)(2Me^{-\omega b}+1) + Me^{-\omega b}\Big)k\Big].$$

Then for $t \in [0, b]$ fixed the set $\{(-A)^{\alpha}\mathcal{H}(x)(t) :\ x \in Z_k\}$ is bounded in $X$. By $(\mathbf{H}_2)$, we deduce that $(-A)^{-\alpha} : X \to X_\alpha$ is compact. It follows that the set $\{\mathcal{H}(x)(t) :\ x \in Z_k\}$ is relatively compact for each $t \in [0, b]$ in $X_\alpha$.

**Step 4 :** The set $\{\mathcal{H}(x) :\ x \in Z_k\}$ is an equicontinuous family of functions.

Let $x \in Z_k$ and $0 \leq \tau_1 < \tau_2 \leq b$, then we have

$$\begin{aligned}
|\mathcal{H}(x)(\tau_2) - \mathcal{H}(x)(\tau_1)|_\alpha \quad \leq \quad & |[C(\tau_2) - C(\tau_1)]\varphi(0)|_\alpha + |[S(\tau_2) - S(\tau_1)]\varphi'(0)|_\alpha \\
& + \Big|\int_0^{\tau_2} S(\tau_2 - s) f(s, \widetilde{x}_s, \widetilde{x'}_s))ds - \int_0^{\tau_1} S(\tau_1 - s) f(s, \widetilde{x}_s, \widetilde{x'}_s))ds\Big|_\alpha \\
\leq \quad & |[C(\tau_2) - C(\tau_1)]\varphi(0)|_\alpha + |[S(\tau_2) - S(\tau_1)]\varphi'(0)|_\alpha + \Big|\int_0^{\tau_1}[S(\tau_2 - s) - S(\tau_1 - s)]f(s, \widetilde{x}_s, \widetilde{x'}_s))ds\Big|_\alpha \\
& + \Big|\int_{\tau_1}^{\tau_2} S(\tau_2 - s) f(s, \widetilde{x}_s, \widetilde{x'}_s))ds\Big|_\alpha \\
\leq \quad & |[C(\tau_2) - C(\tau_1)]\varphi(0)|_\alpha + |[S(\tau_2) - S(\tau_1)]\varphi'(0)|_\alpha + \Big\|(-A)^{\alpha-1}\Big[\int_{\tau_1}^{\tau_2} \frac{d}{ds}\Big(C(\tau_2 - s) f(s, \widetilde{x}_s, \widetilde{x'}_s)\Big)ds - \\
& \int_{\tau_1}^{\tau_2} C(\tau_2 - s)\frac{d}{ds}\Big(f(s, \widetilde{x}_s, \widetilde{x'}_s)\Big)ds\Big]\Big\| + \Big\|(-A)^{\alpha-1}\Big[\int_0^{\tau_1} \frac{d}{ds}\Big([C(\tau_2 - s) - C(\tau_1 - s)]f(s, \widetilde{x}_s, \widetilde{x'}_s)\Big)ds \\
& - \int_0^{\tau_1}[C(\tau_2 - s) - C(\tau_1 - s)]\frac{d}{ds}\Big(f(s, \widetilde{x}_s, \widetilde{x'}_s)\Big)ds\Big]\Big\| \\
\leq \quad & |[C(\tau_2) - C(\tau_1)]\varphi(0)|_\alpha + |[S(\tau_2) - S(\tau_1)]\varphi'(0)|_\alpha \\
& + \|(-A)^{\alpha-1}\|\Big(\|f(\tau_2, \widetilde{x}_{\tau_2}, \widetilde{x'}_{\tau_2}) - C(\tau_2 - \tau_1) f(\tau_1, \widetilde{x}_{\tau_1}, \widetilde{x'}_{\tau_1})\| \\
& + M_1 e^{-\omega b}\|f(\tau_2, \widetilde{x}_{\tau_2}, \widetilde{x'}_{\tau_2}) - f(\tau_1, \widetilde{x}_{\tau_1}, \widetilde{x'}_{\tau_1})\| + \|[C(\tau_2 - \tau_1) - I]f(\tau_1, \widetilde{x}_{\tau_1}, \widetilde{x'}_{\tau_1}\| \\
& + \|[C(\tau_2) - C(\tau_1)]f(0, \widetilde{x}_0, \widetilde{x'}_0)\| + \|C(\tau_2) - C(\tau_1)\|\ \|f(\tau_1, \widetilde{x}_{\tau_1}, \widetilde{x'}_{\tau_1}) - f(0, \widetilde{x}_0, \widetilde{x'}_0)\|\Big) \longrightarrow 0 \ \text{ if } \ \tau_1 \to \tau_2,
\end{aligned}$$

since $(-A)^{\alpha-1}$ is compact from X to X and $(C(t)_{t\in\mathbb{R}}$ is uniformly continuous on compact subsets of X. Thus, $\mathcal{H}$ maps $Z_k$ into an equicontinuous family of functions.
The equicontinuities for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 < 0 < \tau_2$ are obvious.
So from the above **step 1** to **step 4** and the Ascoli-Arzela theorem, we can conclude that $\mathcal{H} : Z_k \to Z_k$ is completely continuous. Hence by the Schauder fixed point theorem, $\mathcal{H}$ has at least one fixed point $\overline{x}$ in $Z_k$ which is a mild solutions of equation (1.1).■

In the following, we prove the uniqueness of local mild solutions of equation (1.1). In what follow, we require $f$ to be autonomous, that is, $f : \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$. We make the following assumption.

($\mathbf{H}_3$) $f$ is locally lipschitz, that is, for each $\delta > 0$ there is a constant $c_0(\delta) > 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{C}_\alpha$ $\|\varphi_1\|_{\mathcal{C}_\alpha}, |\varphi_2\|_{\mathcal{C}_\alpha} \leq \delta$ then

$$\|f(\varphi_1, \varphi_1') - f(\varphi_2, \varphi_2')\| \leq c_0(\delta)\|\varphi_1 - \varphi_2\|_{\mathcal{C}_\alpha}.$$

($\mathbf{H}_4$) The maps $t \to AC(t)$ is locally bounded.

**Theorem 3.5.** *Assume that* ($\boldsymbol{H_0}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H}_3$) *and* ($\boldsymbol{H_4}$) *hold. Let* $\varphi \in \mathcal{C}_\alpha$ *such that* $\varphi(0) \in D(A)$ *and* $\varphi'(0) \in E$. *Then, there exists a maximal interval of existence* $[-r, b_\varphi[$ *and a unique mild solution* $u(., \varphi)$ *of equation* (1.1) *defined on* $[-r, b_\varphi[$ *and either*

$$b_\varphi = +\infty \text{ or } \overline{\lim}_{t \to b_\varphi^-}(|u(t, \varphi)|_\alpha + |u'(t, \varphi)|_\alpha) = +\infty.$$

*Moreover,* $u(t, \varphi)$ *is a continuous function of* $\varphi$ *in the sense that if* $\varphi \in \mathcal{C}_\alpha$ *and* $t \in [0, b_\varphi[$, *then there exist positive constants* $k$ *and* $\varepsilon$ *such that, for* $\varphi, \psi \in \mathcal{C}_\alpha$ *and* $\|\varphi - \psi\|_{\mathcal{C}_\alpha} < \varepsilon$, *we have*

$$t \in [0, b_\psi[ \text{ and } |u(s, \varphi) - u(s, \psi)|_\alpha + |u'(s, \varphi) - u'(s, \psi)|_\alpha \leq k\|\varphi - \psi\|_{\mathcal{C}_\alpha} \text{ for all } s \in [-r, t].$$

**Proof**. Let $b_1 > 0$. The local lipschitz condition on $f$ implies that for each $\alpha > 0$, there exists $c_0(\delta)$ such that for $\varphi \in \mathcal{C}$ with $\|\varphi\|_{\mathcal{C}_\alpha} < \delta$ we have

$$\|f(\varphi, \varphi')\| \leq c_0(\delta)\|\varphi\|_{\mathcal{C}_\alpha} + \|f(0, 0)\| \leq c_0(\delta)\delta + \|f(0, 0)\|,$$

with for given $\varphi \in \mathcal{C}_\alpha$, $\delta = \|\varphi\|_{\mathcal{C}_\alpha} + 1$ and $c_1(\delta) = c_0(\delta)\delta + \sup\limits_{s \in [0, b_1]} \|f(0, 0)\|$. Consider the following set

$$Z_\varphi = \left\{ \begin{array}{l} u \in C^1([-r, b_1]; X_\alpha) : u(s) = \varphi(s),\ u'(s) = \varphi'(s) \text{ if } s \in [-r, 0] \\ \\ \text{and } \sup_{s \in [0, b_1]}(|u(s) - \varphi(0)|_\alpha + |u'(s) - \varphi'(0)|_\alpha) \leq 1, \end{array} \right\}$$

then $Z_\varphi$ is a closed set of $C^1([-r, b_1]; X_\alpha)]$. Consider the mapping

$$\mathcal{K} : Z_\varphi \to C^1([-r, b_1]; X_\alpha)$$

defined by

$$\left\{ \begin{array}{l} \mathcal{K}(u)(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \displaystyle\int_0^t S(t - s)f(u_s, u_s')ds \text{ for } t \geq 0 \\ \\ \mathcal{K}(u_0)(t) = \varphi(t) \ \text{ for } t \in [-r, 0] \\ \\ (\mathcal{K}(u_0))'(t) = \varphi'(t) \ \text{ for } t \in [-r, 0]. \end{array} \right.$$

We will show that $\mathcal{K}(Z_\varphi) \subset Z_\varphi$. Let $u \in Z_\varphi$, $t \in [0, b_1]$ and $\mu_0$ be a positive real number such that $\|AC(t)\| \le \mu_0$ for all $t \in [0, b_1]$. Then we have

$$\begin{aligned}
|\mathcal{K}(u)(t) - \varphi(0)|_\alpha &\le |C(t)\varphi(0) - \varphi(0)|_\alpha + |S(t)\varphi'(0)|_\alpha + \Big| \int_0^t S(t-s) f(u_s, u'_s) ds \Big|_\alpha \\
&\le |C(t)\varphi(0) - \varphi(0)|_\alpha + |S(t)\varphi'(0)|_\alpha + \Big\| (-A)^{\alpha-1} \int_0^t \Big( \int_0^{t-s} AC(\sigma) f(u_s, u'_s) d\sigma \Big) ds \Big\| \\
&\le |C(t)\varphi(0) - \varphi(0)|_\alpha + |S(t)\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \mu_0 b_1 \int_0^t \|f(u_s, u'_s)\| ds.
\end{aligned}$$

Since $|u(s) - \varphi(0)|_\alpha + |u'(s) - \varphi'(0)|_\alpha \le 1$ for $s \in [0, b_1]$ and $\delta = \|\varphi\|_{\mathcal{C}_\alpha} + 1$, we deduce that $\|u_s\|_{\mathcal{C}_\alpha} \le 1 + \|\varphi\|_{\mathcal{C}_\alpha} = \delta$ for $s \in [0, b_1]$. Then

$$\begin{aligned}
\|f(u_s, u'_s)\| &\le c_0(\delta)\|u_s\|_{\mathcal{C}_\alpha} + \|f(0,0)\| \le c_1(\delta) \\
\|f(\varphi, \varphi')\| &\le c_0(\delta)\|\varphi\|_{\mathcal{C}_\alpha} + \|f(0,0)\| \le c_1(\delta).
\end{aligned} \tag{3.2}$$

If we choose $b_1$ sufficiently small such that

$$\sup_{s \in [0, b_1]} \Big\{ |C(s)\varphi(0) - \varphi(0)|_\alpha + |S(s)\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \mu_0 b_1 c_1(\delta) s \Big\} < \frac{1}{2},$$

consequently

$$|\mathcal{K}(u)(t) - \varphi(0)|_\alpha \le |C(t)\varphi(0) - \varphi(0)|_\alpha + |S(t)\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \mu_0 b_1^2 c_1(\delta) < \frac{1}{2} \text{ for } t \in [0, b_1].$$

On the other hand using equation (2.1) and Proposition 2.3 we have

$$\begin{aligned}
(\mathcal{K}(u))'(t) &= C'(t)\varphi(0) + S'(t)\varphi'(0) + \int_0^t C(t-s) f(u_s, u'_s) ds \text{ for } t \ge 0 \\
|(\mathcal{K}(u))'(t) - \varphi'(0)|_\alpha &\le |AS(t)\varphi(0)|_\alpha + |C(t)\varphi'(0) - \varphi'(0)|_\alpha + \Big\| (-A)^{\alpha-1} \int_0^t AC(t-s) f(u_s, u'_s) ds \Big\| \\
&\le |AS(t)\varphi(0)|_\alpha + |C(t)\varphi'(0) - \varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \, \mu_0 \sup_{s \in [0, b_1]} \|f(u_s, u'_s)\| t \\
&\le |AS(t)\varphi(0)|_\alpha + |C(t)\varphi'(0) - \varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \, \mu_0 c_1(\delta) t.
\end{aligned}$$

We also choose $b_1$ sufficiently small such that

$$\sup_{s \in [0, b_1]} \Big\{ |AS(s)\varphi(0)|_\alpha + |C(s)\varphi'(0) - \varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \, \mu_0 c_1(\delta) s \Big\} < \frac{1}{2},$$

consequently

$$|(\mathcal{K}(u))'(t) - \varphi'(0)|_\alpha < \frac{1}{2}.$$

Finally we have

$$|\mathcal{K}(u)(t)-\varphi(0)|_\alpha+|(\mathcal{K}(u))'(t)-\varphi'(0)|_\alpha<1,$$

hence $K(Z_\varphi)\subset Z_\varphi$.

Let $u,v\in Z_\varphi$ and $t\in[0,b_1]$. Then we have

$$\begin{aligned}
|\mathcal{K}(u)(t)-\mathcal{K}(v)(t)|_\alpha &= \Big|\int_0^t S(t-s)(f(u_s,u'_s)-f(v_s,v'_s))ds\Big|_\alpha\\
&\leq \Big\|-(-A)^{\alpha-1}\int_0^t\Big(\int_0^{t-s}AC(\sigma)[f(u_s,u'_s)-f(v_s,v'_s)]d\sigma\Big)ds\Big\|\\
&\leq \|(-A)^{\alpha-1}\|\mu_0 b_1\int_0^t\|f(u_s,u'_s)-f(v_s,v'_s)\|ds\\
&\leq \|(-A)^{\alpha-1}\|\mu_0 b_1^2 c_0(\delta)\|u-v\|_{\mathcal{C}_\alpha}.
\end{aligned}$$

Since

$$\sup_{s\in[0,b_1]}\Big\{|C(s)\varphi(0)-\varphi(0)|_\alpha+|S(s)\varphi'(0)|_\alpha+\|(-A)^{\alpha-1}\|\mu_0 b_1 c_1(\delta)s\Big\}<\frac{1}{2},$$

it follows that

$$|\mathcal{K}(u)(t)-\mathcal{K}(v)(t)|_\alpha<\frac{1}{2}\|u-v\||_{\mathcal{C}_\alpha}.$$

Using the same reasoning like previously, we have

$$\begin{aligned}
|(\mathcal{K}(u))'(t)-(\mathcal{K}(v))'(t)|_\alpha &= \Big|\int_0^t C(t-s)(f(u_s,u'_s)-f(s,v_s,v'_s))ds\Big|_\alpha\\
&< \frac{1}{2}\|u-v\||_{\mathcal{C}_\alpha}.
\end{aligned}$$

Adding the two previous equations

$$\|(\mathcal{K}(u))-(\mathcal{K}(v))\|_{\mathcal{C}_\alpha} < \|u-v\||_{\mathcal{C}_\alpha},$$

it follows that $\mathcal{K}$ is a strict contraction in $Z_\varphi$. Thus by a fixed point theorem, $\mathcal{K}$ has a unique fixed point $u$ in $Z_\varphi$.
Let $\widehat{u}$ an other mild solution of equation (1.1) on $Z_\varphi$ corresponding to $\varphi$. Then we have

$$\begin{aligned}
\|u-\widehat{u}\|_{\mathcal{C}_\alpha} &= \|(\mathcal{K}(u))-(\mathcal{K}(\widehat{u}))\|_{\mathcal{C}_\alpha}\\
&< \|u-\widehat{u}\||_{\mathcal{C}_\alpha},
\end{aligned}$$

which gives a contradiction. We conclude that equation (1.1) has one and only one mild solution which is defined on $[-r,b_1]$ and denoted by $u(.,\varphi)$. Using the same arguments, we can show that $u(.,\varphi)$ can be extended to a maximal interval of existence $[0,b_\varphi[$. If we assume that $b_\varphi<+\infty$ and $\overline{\lim}_{t\to b_\varphi^-}(|u(t,\varphi)|_\alpha+|u'(t,\varphi)|_\alpha)<+\infty$, then there exists a constant $\delta>0$ such that $(|u(t,\varphi)|_\alpha+$

$|u'(t,\varphi)|_\alpha) \leq \alpha$ for all $t \in [0, b_\varphi[$. We claim that $u(.,\varphi)$ and $u'(.,\varphi)$ are uniformly continuous. Consequently

$$\lim_{t\to b_\varphi^-} (u(t,\varphi) + u'(t,\varphi)) \text{ exists,}$$

which contradicts the maximality of $[0, b_\varphi[$. Let us show the uniform continuity of $u(.,\varphi)$ and $u'(.,\varphi)$. Let $t,\ t+h \in [0, b_\varphi[$, $h > 0$ and $\theta \in [-r, 0]$. If $t+\theta \geq 0$, then the map $t \mapsto C(t+\theta)\varphi(0)+S(t+\theta)\varphi'(0)$ is uniformly continuous. On the other hand let $\mu$ be a positive number such that $\|AC(t\| \leq \mu$ for all $t \in [0, b_\varphi[$ and pose $u(.,\varphi) = u$. We have

$$\begin{aligned}
u(t+h+\theta) - u(t+\theta) &= C(t+h+\theta)\varphi(0) - C(t+\theta)\varphi(0) + S(t+h+\theta)\varphi'(0) - S(t+\theta)\varphi'(0) \\
&\quad + \int_0^{t+\theta+h} S(t+\theta+h-s) f(u_s, u'_s)ds - \int_0^{t+\theta} S(t+\theta-s) f(u_s, u'_s)ds \\
&= C(t+h+\theta)\varphi(0) - C(t+\theta)\varphi(0) + S(t+h+\theta)\varphi'(0) - S(t+\theta)\varphi'(0) \\
&\quad + \int_0^{t+\theta} S(s) f(u_{t+\theta+h-s}, u'_{t+\theta+h-s})ds + \int_{t+\theta}^{t+\theta+h} S(s) f(u_{t+\theta+h-s}, u'_{t+\theta+h-s})ds \\
&= C(t+h+\theta)\varphi(0) - C(t+\theta)\varphi(0) + S(t+h+\theta)\varphi'(0) - S(t+\theta)\varphi'(0) \\
&\quad + \int_0^{t+\theta} S(s) \Big[ f(u_{t+\theta+h-s}, u'_{t+\theta+h-s}) - f(u_{t+\theta-s}, u'_{t+\theta-s}) \Big] ds \\
&\quad + \int_{t+\theta}^{t+\theta+h} S(s) f(u_{t+\theta+h-s}, u'_{t+\theta+h-s})ds.
\end{aligned}$$

Thus, using the local Lipschitz condition of $f$, we have

$$\begin{aligned}
|u(t+h+\theta,\varphi) - u(t+\theta,\varphi)|_\alpha &\leq |C(t+h+\theta)\varphi(0) - C(t+\theta)\varphi(0)|_\alpha + |S(t+h+\theta)\varphi'(0) - S(t+\theta)\varphi'(0)|_\alpha \\
&\quad + \Big\| (-A)^{\alpha-1} \int_0^{t+\theta} \Big( \int_0^{t-s} AC(\sigma) \Big[ f(u_{t+\theta+h-s}, u'_{t+\theta+h-s}) - f(u_{t+\theta-s}, u'_{t+\theta-s}) \Big] d\sigma \Big) ds \Big\| \\
&\quad + \Big\| (-A)^{\alpha-1} \int_{t+\theta}^{t+\theta+h} \Big( \int_0^{t-s} AC(\sigma) f(u_{t+\theta+h-s}, u'_{t+\theta+h-s}) d\sigma \Big) ds \Big\| \\
&\leq |C(t+h+\theta)\varphi(0) - C(t+\theta)\varphi(0)|_\alpha + |S(t+h+\theta)\varphi'(0) - S(t+\theta)\varphi'(0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\| \mu b_\varphi \int_0^{t+\theta} \| f(u_{t+\theta+h-s}, u'_{t+\theta+h-s}) - f(u_{t+\theta-s}, u'_{t+\theta-s}) \| ds \\
&\quad + \|(-A)^{\alpha-1}\| \mu b_\varphi \int_{t+\theta}^{t+\theta+h} \| f(u_{t+\theta+h-s}, u'_{t+\theta+h-s}) \| ds \\
&\leq |C(t+h+\theta)\varphi(0) - C(t+\theta)\varphi(0)|_\alpha + |S(t+h+\theta)\varphi'(0) - S(t+\theta)\varphi'(0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\| \mu b_\varphi c_1(\delta) h + \|(-A)^{\alpha-1}\| \mu b_\varphi \int_0^t \|u_{s+h} - u_s\|_{\mathcal{C}_\alpha} ds
\end{aligned}$$

If $t+\theta < 0$. Let $h_0 > 0$ sufficiently small such for $h \in ]0, h_0[$

$$|u_{t+h}(\theta) - u_t(\theta)|_\alpha \quad \leq \quad \sup_{-r\leq\sigma\leq 0} |u(\sigma+h) - u(\sigma)|_\alpha = \|u_h - \varphi\|_\alpha$$

Since the map $t \mapsto C(t)\varphi(0) + S(t)\varphi'(0)$ is uniformly continuous, consequently, for $t, t+h \in [0, b_\varphi[$ and $h \in ]0, h_0[$, we have

$$\|u_{t+h}(.) - u_t(.)\|_\alpha \leq \delta_1(h) + \delta_2(h) + \|(-A)^{\alpha-1}\|\mu b_\varphi c_1(\delta)h + \|(-A)^{\alpha-1}\|\mu b_\varphi c_0(\delta)\int_0^t \|u_{s+h} - u_s\|_{\mathcal{C}_\alpha} ds$$

where

$$\delta_1(h) = \|u_h - \varphi\|_\alpha \text{ and } \delta_2(h) = \sup_{\substack{t \\ t+h \in [0,b_\varphi[}} \Big(|C(t+h)\varphi(0) - C(t)\varphi(0)|_\alpha + |S(t+h)\varphi'(0) - S(t)\varphi'(0)|_\alpha\Big).$$

From [11] (in Proposition 2.4), $t \to C(t)\varphi(0) + S(t)\varphi'(0)$ belongs to $C^2([0, b_\varphi]; X)$, by a similar reasoning, we also have

$$\|u'_{t+h}(.) - u'_t(.)\|_\alpha \leq \delta'_1(h) + \delta'_2(h) + \|(-A)^{\alpha-1}\|\mu c_1(\delta)h + \|(-A)^{\alpha-1}\|\mu c_0(\delta)\int_0^t \|u_{s+h}(.) - u_s(.)\|_{\mathcal{C}_\alpha} ds$$

where

$$\delta'_1(h) = \|u'_h - \varphi'\|_\alpha \text{ and } \delta'_2(h) = \sup_{\substack{t \\ t+h \in [0,b_\varphi[}} \Big(|AS(t+h)\varphi'(0) - AS(t)\varphi'(0)|_\alpha + |C(t+h)\varphi'(0) - C(t)\varphi'(0)|_\alpha\Big).$$

Adding the previous inequality, we have

$$\|u_{t+h}(.) - u_t(.)\|_{\mathcal{C}_\alpha} \leq \gamma(h) + \|(-A)^{\alpha-1}\|\mu c_0(\delta)(1 + b_\varphi)\int_0^t \|u_{s+h} - u_s\|_{\mathcal{C}_\alpha} ds,$$

with

$$\gamma(h) = \delta_1(h) + \delta_2(h) + \delta'_1(h) + \delta'_2(h) + \|(-A)^{\alpha-1}\|\mu c_1(\delta)(1 + b_\varphi)h.$$

By Gronwall's lemma, it follows that

$$\|u_{t+h}(., \varphi) - u_t(., \varphi)\|_{\mathcal{C}_\alpha} \leq \gamma(h)\exp[\|(-A)^{\alpha-1}\|\mu c_0(\delta)(1 + b_\varphi)b_\varphi].$$

This completes that $u$ and $u'$ are uniformly continuous and $u$ can be extended over $[0, b_\varphi + \eta]$, which contradicts the maximality of $[0, b_\varphi[$. Using the same reasoning, one can show a similar result for $h < 0$.
Now, we want to prove that the solution depends continuously on initial data. Let $\varphi \in \mathcal{C}_\alpha$ and $t \in [0, b_\varphi[$ be fixed. Set

$$\delta = 1 + \sup_{-r \leq s \leq t} \|u_s(., \varphi)\|_{\mathcal{C}_\alpha}$$

and

$$c(t) = \Big(2M_1 e^{-\omega t} + \mu b_\varphi\Big)\exp\Big(\|(-A)^{\alpha-1}\|c_0(\delta)\mu(1 + b_\varphi)t\Big).$$

Let $\varepsilon \in ]0, 1[$ and $\psi \in \mathcal{C}_\alpha$ such that $\|\varphi - \psi\|_{\mathcal{C}_\alpha} < \varepsilon$. Then

$$\|\psi\|_{\mathcal{C}_\alpha} \leq \|\varphi\|_{\mathcal{C}_\alpha} + \varepsilon < \delta.$$

We define

$$b_0 := \sup\{s > 0 : \|u_\sigma(., \psi)\|_{\mathcal{C}_\alpha} \leq \delta \text{ for } \sigma \in [0, s]\}.$$

If we suppose that $b_0 < t$, we obtain for $s \in [0, b_0]$

$$\begin{aligned}
\|u_s(.,\varphi) - u_s(.,\psi)\|_\alpha &\leq M_1 e^{-\omega s}\|\varphi - \psi\|_\alpha + M_1 e^{-\omega s}\|\varphi' - \psi'\|_\alpha \\
&+ \Big\|(-A)^{\alpha-1}\int_0^s \Big(\int_0^{s-\sigma} AC(\xi)[f(u_\sigma(.,\varphi), u'_\sigma(.,\varphi)) - f(u_\sigma(.,\psi), u'_\sigma(.,\psi))]d\xi\Big)d\sigma\Big\| \\
&\leq M_1 e^{-\omega s}\|\varphi - \psi\|_\alpha + M_1 e^{-\omega s}\|\varphi' - \psi'\|_\alpha \\
&+ \|(-A)^{\alpha-1}\|\mu b_\varphi \int_0^s \|f(u_\sigma(.,\varphi), u'_\sigma(.,\varphi)) - f(u_\sigma(.,\psi), u'_\sigma(.,\psi))\|d\sigma \\
&\leq M_1 e^{-\omega s}\|\varphi - \psi\|_\alpha + M_1 e^{-\omega s}\|\varphi' - \psi'\|_\alpha + \|(-A)^{\alpha-1}\|\mu b_\varphi c_0(\delta)\int_0^s \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\|_{\mathcal{C}_\alpha} ds.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\|u'_s(.,\varphi) - u'_s(.,\psi)\|_\alpha &\leq \Big\|\int_0^s AC(\sigma)[\varphi(0) - \psi(0)]d\sigma\Big\|_\alpha + M_1 e^{-\omega s}\|\varphi' - \psi'\|_\alpha + \|(-A)^{\alpha-1}\|c_0(\delta)\mu\int_0^s \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\|_{\mathcal{C}_\alpha} ds \\
&\leq \mu b_\varphi\|\varphi - \psi\|_{\mathcal{C}_\alpha} + M_1 e^{-\omega t}\|\varphi' - \psi'\|_\alpha + \|(-A)^{\alpha-1}\|c_0(\delta)\mu\int_0^s \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\|_{\mathcal{C}_\alpha} ds.
\end{aligned}$$

By adding the previous inequality, we have

$$\|u_s(.,\varphi) - u_s(.,\psi)\|_{\mathcal{C}_\alpha} \leq \Big(2M_1 e^{-\omega t} + \mu b_\varphi\Big)\|\varphi - \psi\|_{\mathcal{C}_\alpha} + \|(-A)^{\alpha-1}\|c_0(\delta)\mu(1 + b_\varphi)\int_0^s \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\|_{\mathcal{C}_\alpha} ds.$$

By Gronwall's lemma, we deduce that

$$\|u_s(.,\varphi) - u_s(.,\psi)\|_{\mathcal{C}_\alpha} \leq c(t)\|\varphi - \psi\|_{\mathcal{C}_\alpha}. \tag{3.3}$$

This implies that

$$\|u_s(.,\psi)\|_{\mathcal{C}_\alpha} \leq c(t)\varepsilon + \delta - 1 < \delta \text{ for all } s \in [0, b_0].$$

It follows that $b_0$ cannot be the largest number $s > 0$ such that $\|u_s(.,\psi)\|_{\mathcal{C}_\alpha} < \delta$, for $\sigma \in [0, s]$. Thus $b_0 \geq t$ and $t < b_\psi$. Furthermore, $\|u_s(.,\varphi)\|_{\mathcal{C}_\alpha} < \delta$ for $s \in [0, t]$, then using the inequality (3.3) we deduce the continuous dependence on the initial data.■

**Corollary 3.6.** *Assume that (**$H_0$**), (**$H_2$**), (**$H_3$**) and (**$H_4$**) hold. Let $\varphi \in \mathcal{C}_\alpha$ such that $\varphi(0) \in D(A)$ and $\varphi'(0) \in E$. Let $k_1$ be a continuous function on $\mathbb{R}^+$ and $k_2 \in L^1_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ be such that $\|f(\varphi,\varphi')\| \leq k_1(t)\|\varphi\|_{\mathcal{C}_\alpha} + k_2(t)$ for $t \geq 0$ and $\varphi, \varphi' \in \mathcal{C}_\alpha$. Then equation* (1.1) *has a unique mild solution which is defined for all $t \geq 0$.*

**Proof**. Let $[-r, b_\varphi[$ denote the maximal interval of existence of the mild solution $u(t,\varphi)$ of equation (1.1). Then

$$b_\varphi = +\infty \text{ or } \overline{\lim}_{t\to b_\varphi^-}(|u(t,\varphi)|_\alpha + |u'(t,\varphi)|_\alpha) = +\infty.$$

If $b_\varphi < +\infty$, then $\overline{\lim}_{t\to b_\varphi^-}(|u(t,\varphi)|_\alpha + |u'(t,\varphi)|_\alpha) = +\infty$. Thus, we have

$$\begin{aligned}
|u(t,\varphi)|_\alpha &\leq |C(t)\varphi(0)|_\alpha + |S(t)\varphi'(0)|_\alpha + \Big|\int_0^t S(t-s)f(u_s,u_s')ds\Big|_\alpha \\
&\leq M_1e^{-\omega b_\varphi}(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha) + \Big\|(-A)^{\alpha-1}\int_0^s\Big(\int_0^{t-s}AC(\xi)f(u_s(.,\varphi),u_s'(.,\varphi))d\xi\Big)ds\Big\| \\
&\leq k_0 + \|(-A)^{\alpha-1}\|\mu b_\varphi\int_0^t k_1(s)\|u_s\|_{\mathcal{C}_\alpha}ds \text{ for } t\in[0,b_\varphi[,
\end{aligned}$$

where

$$k_0 = (2M_1e^{-\omega b_\varphi} + \mu b_\varphi)(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha) + \|(-A)^{\alpha-1}\|\mu(b_\varphi+1)\int_0^{b_\varphi}k_2(s)ds\Big].$$

On the other hand, we have

$$\begin{aligned}
|u'(t,\varphi)|_\alpha &\leq \mu b_\varphi|\varphi'(0)|_\alpha + M_1e^{-\omega b_\varphi}|\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\|\mu\int_0^t\|f(u_s,u_s')\|ds \\
&\leq k_0 + \|(-A)^{\alpha-1}\|\mu\int_0^t k_1(s)\|u_s\|_{\mathcal{C}_\alpha}ds \text{ for } t\in[0,b_\varphi[.
\end{aligned}$$

By Gronwall's lemma, we deduce that

$$\|u_t(\varphi)\|_{\mathcal{C}_\alpha} \leq 2k_0\exp\Big(\|(-A)^{\alpha-1}\mu(b_\varphi+1)\int_0^t k_1(s)\Big)ds < +\infty \text{ for } t\in[0,b_\varphi[,$$

and

$$\overline{\lim}_{t\to b_\varphi^-}(|u(t,\varphi)|_\alpha + |u'(t,\varphi)|_\alpha) < +\infty,$$

which gives a contradiction.■

As an immediat consequence, we get the following result.

**Corollary 3.7.** *Assume that* ($\boldsymbol{H_0}$) *and* ($\boldsymbol{H_2}$) *hold and there exists a positive constant* $L$ *such that for* $\varphi_1,\varphi_2\in\mathcal{C}_\alpha$

$$\|f(\varphi_1,\varphi_1') - f(\varphi_2,\varphi_2')\| \leq L\|\varphi_1-\varphi_2\|_{\mathcal{C}_\alpha} \textit{ for } t\geq 0.$$

*Let* $\varphi\in\mathcal{C}_\alpha$ *such that* $\varphi(0)\in D(A)$ *and* $\varphi'(0)\in E$. *Then equation* (1.1) *has a unique mild solution which is defined for all* $t\geq 0$.

# 4 Existence of strict solutions

**Theorem 4.1.** *Assume that* ($\boldsymbol{H_0}$) *and* ($\boldsymbol{H_2}$) *hold and* $f$ *is continuously differentiable. Moreover assume that the partial derivatives* $D_1f$ *and* $D_2f$ *are locally Lipschitz in the classical sense. Let* $\varphi$ *be in* $C^3([-r,0],D((-A)^\alpha)$ *such that* $\varphi(0),\varphi''(0)\in D(A)$, $\varphi'(0),\varphi^{(3)}(0)\in E$, $\varphi''(0) = A\varphi(0)+f(\varphi,\varphi')$ *and* $\varphi^{(3)}(0) = A\varphi'(0)$. *Then the corresponding mild solution* $u$ *is a strict solution of equation* (1.1).

**Proof**. Let $\varphi$ be in $C^3([-r,0],X)$ such that $\varphi(0),\varphi''(0)\in D(A)$, $\varphi'(0),\varphi^{(3)}(0)\in E$, $\varphi''(0)=A\varphi(0)+f(\varphi,\varphi')$ and $\varphi^{(3)}(0)=A\varphi'(0)$. Let $u$ be the corresponding mild solution of equation (1.1) which is defined on some maximal interval $[0,b_\varphi[$ and let $a<b_\varphi$. Then by using the strict contraction principle, we can show that there exists a unique continuous function $v$ such that

$$v(t)=\begin{cases} C(t)(A\varphi(0)+f(\varphi,\varphi'))+S(t)A\varphi'(0)+\displaystyle\int_0^t C(t-s)[D_1f(u_s,u'_s)u'_s+D_2f(u_s,u'_s)v_sds] \\ \\ v_0=\varphi''. \end{cases}$$

We introduce the function $w$ defined by

$$\begin{cases} w(t)=\varphi'(0)+\displaystyle\int_0^t v(s)ds \text{ if } t\geq 0 \\ \\ w(t)=\varphi'(t) \text{ if } -r\leq t\leq 0 \\ \\ w'(t)=\varphi''(t) \text{ if } -r\leq t\leq 0. \end{cases}$$

We will show that $w=u'$. We can also see that

$$w_t=\varphi'+\int_0^t v_sds \ \text{ for } t\in[0,a].$$

Consequently, the maps $t\to w_t$ and $t\to\displaystyle\int_0^t C(t-s)f(u_s,w_s)ds$ are continuously differentiable and the following formula holds

$$\begin{aligned} \frac{d}{dt}\int_0^t C(t-s)f(u_s,w_s)ds &= \frac{d}{dt}\int_0^t C(s)f(u_{t-s},w_{t-s})ds \\ &= C(t)f(\varphi,\varphi')+\int_0^t C(t-s)[D_1f(u_s,w_s)u'_s+D_2f(u_s,w_s)w'_sds] \\ &= C(t)f(\varphi,\varphi')+\int_0^t C(t-s)[D_1f(u_s,w_s)u'_s+D_2f(u_s,w_s)v_sds], \end{aligned}$$

which implies

$$\int_0^t C(s)f(\varphi,\varphi')ds = \int_0^t C(t-s)f(u_s,w_s)ds-\int_0^t C(t-s)[D_1f(u_s,w_s)u'_s+D_2f(u_s,w_s)v_sds].$$

Consequently we have

$$\begin{aligned} w(t) &= \varphi'(0)+\int_0^t C(s)A\varphi(0)\,ds+\int_0^t C(t-s)f(u_s,w_s)ds+\int_0^t S(s)A\varphi'(0)\,ds \\ &\quad -\int_0^t\int_0^s C(s-\tau)[D_1f(u_\tau,w_\tau)u'_\tau+D_2f(u_\tau,w_\tau)v_\tau]d\tau ds+\int_0^t\int_0^s C(s-\tau)[D_1f(u_\tau,u'_\tau)u'_\tau+D_2f(u_\tau,u'_\tau)v_\tau]d\tau ds. \end{aligned}$$

Since by equation (2.1) and Proposition 2.3, we have

$$\begin{aligned}\int_0^t C(s)A\varphi(0)ds &= S(t)A\varphi(0)\\ \int_0^t S(s)A\varphi'(0)\,ds &= C(t)\varphi'(0)-\varphi'(0),\end{aligned}$$

it follows that

$$\begin{aligned}w(t) &= S(t)A\varphi(0)+\int_0^t C(t-s)f(u_s,w_s)ds+C(t)\varphi'(0)\\ &\quad+\int_0^t\int_0^s C(s-\tau)\Big[D_1f(u_\tau,u'_\tau)u'_\tau-D_1f(u_\tau,w_\tau)u'_\tau\Big]d\tau ds\\ &\quad+\int_0^t\int_0^s C(s-\tau)\Big[D_2f(u_\tau,u'_\tau)v_\tau-D_2f(u_\tau,w_\tau)v_\tau\Big]d\tau ds.\end{aligned}$$

Since for $t\geq 0$, we have

$$u'(t)=AS(t)\varphi(0)+C(t)\varphi'(0)+\int_0^t C(t-s)f(u_s,u'_s)ds,$$

then for $t\in[0,a]$, we have

$$\begin{aligned}|u'(t)-w(t)|_\alpha &\leq \Big|\int_0^t C(t-s)[f(u_s,u'_s)-f(u_s,w_s)]ds\Big|_\alpha+\Big|\int_0^t\int_0^s C(s-\tau)\Big[D_1f(u_\tau,u'_\tau)u'_\tau-D_1f(u_\tau,w_\tau)u'_\tau\Big]d\tau ds\Big|\\ &\quad+\Big|\int_0^t\int_0^s C(s-\tau)\Big[D_2f(u_\tau,u'_\tau)v_\tau-D_2f(u_\tau,w_\tau)v_\tau\Big]d\tau ds\Big|. \qquad (4.1)\end{aligned}$$

Let $H=\{u'_s,w_s: s\in[0,a]\}$. Then $H$ is a compact set, it follows that $f$, $D_1f$ and $D_2f$ are globally lipschitz on $H$. Let $c_1$ be such that for $t\in[0,a]$ and $x,y\in H$, we have

$$\begin{aligned}\|f(x,x')-f(y,y')\| &\leq c_1\|x-y\|_{\mathcal{C}_\alpha}\\ \|D_1f(x,x')-D_1f(y,y')\| &\leq c_1\|x-y\|_{\mathcal{C}_\alpha}\\ \|D_2f(x,x')-D_2f(y,y')\| &\leq c_1\|x-y\|_{\mathcal{C}_\alpha}.\end{aligned}$$

Consequently, using equation (4.1) we can find a positive constant $k(a)$ such that by Gronwall's lemma,

$$\|u'_\tau-w_\tau\|_\alpha \leq k(a)\int_0^t\|u'_s-w_s\|_{\mathcal{C}_\alpha}ds \text{ for } s\in[0,a],$$

which implies that $u'=w$. Consequently, we deduce that the mild solution is twice continuously differentiable from $[-r,a]$ to $X$. We deduce that $u$ is a strict solution of equation (1.1) on $[0,a]$. This holds for any $a<b_\varphi$.■

# 5 Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \dfrac{\partial^2 z(t,\xi)}{\partial t^2} = \dfrac{\partial^2 z(t,\xi)}{\partial x^2} + g\Big(t, \dfrac{\partial}{\partial x}[z(t+\theta,\xi)], \dfrac{\partial}{\partial x}[z'(t+\theta,\xi)]\Big) \text{ for } t \geq 0 \text{ and } \xi \in [0,\pi] \\ z(t,0) = z(t,\pi) = 0 \text{ for } t \geq 0 \\ z(\theta,\xi) = \varphi_0(\theta,\xi) \text{ for } \theta \in [-r,0] \text{ and } \xi \in [0,\pi], \end{cases} \tag{5.1}$$

where $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant L such that for $x, y, x_1, y_1 \in \mathbb{R}$

$$|g(t,x,y) - g(t,x_1,y_1)| \leq L(|x-x_1| + |y-y_1|).$$

For example, we can take $g(t,x,y) = e^{-t^2}\left[\sin\left(\frac{x}{2}\right) + \sin\left(\frac{y}{2}\right)\right]$ for $(t,x,y) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$. We can see that $|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq \frac{1}{2}(|x_1-x_2| + |y_1-y_2|)$. The function $\varphi_0 : [-r,0]\times[0,\pi] \to \mathbb{R}$ can be defined by $\varphi_0(\theta,\xi) = e^{-\theta}\sin\xi$. To rewrite equation (5.1) in the abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$, functions vanishing at 0 and $\pi$, equipped with the $L^2$ norm that is to say for all $u \in X$,

$$|u|_{L^2} = \Big(\int_0^\pi |u(x)|^2 dx\Big)^{\frac{1}{2}}.$$

Let $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx),\ x \in [0,\pi],\ n \in \mathbb{N}^*$, then $(e_n)_{n\in\mathbb{N}^*}$ is an orthonormal base for $X$. Let $A : X \to X$ be defined by

$$\begin{cases} Ay = y'' \\ D(A) = \Big\{y \in X : \text{ y, y' are absolutely continuous, } \ y'' \in X,\ \ y(0) = y(\pi) = 0\Big\}, \end{cases}$$

then

$$Ay = \sum_{n=1}^{+\infty} -n^2(y,e_n)e_n,\ y \in D(A),$$

where

$$(g,h) = \int_0^\pi g(s)h(s)ds, \quad \text{for } \ g,h \in X.$$

It is well known that $A$ is the infinitesimal generator of strongly continuous cosine family $C(t),\ t \in \mathbb{R}$ in $X$ given by

$$C(t)y = \sum_{n=1}^{+\infty} \cos nt(y,e_n)e_n,\ y \in X,$$

and that the associated sine family is given by

$$S(t)y = \sum_{n=1}^{+\infty} \frac{1}{n}\sin(nt)(y,e_n)e_n,\ y \in X.$$

If we choose $\alpha = \dfrac{1}{2}$, then ($\mathbf{H_0}$) and ($\mathbf{A_1}$) are satisfied since

$$(-A)^{\frac{1}{2}} y = \sum_{n=1}^{+\infty} n(y, e_n)e_n,\ y \in D\Big((-A)^{\frac{1}{2}}\Big)$$

and

$$(-A)^{\frac{-1}{2}} y = \sum_{n=1}^{+\infty} \frac{1}{n}(y, e_n)e_n,\ y \in X.$$

From [10], the compactness $A^{-1}$ follows from Lemma 2.6, and the fact that the eigenvalues of $(-A)^{\frac{-1}{2}}$ are $\lambda_n = \dfrac{1}{n}$, $\ n = 1, 2, ...$, then ($\mathbf{H_2}$) is satisfied.

We define the phase space

$$\mathcal{C} = C^1([-r, 0]; X)$$

where $C^1([-r, 0]; X)$ is the space of bounded uniformly continuous differentiable functions from $[-r, 0]$ into $X$ with the norm $|\varphi| = \sup\limits_{-r\leq\theta\leq 0} |\varphi(\theta)|$ and let $f : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}} \times \mathcal{C}_{\frac{1}{2}} \to X$ be defined by

$$f(t, \varphi, \varphi')(x) = g\Big(t, \frac{\partial}{\partial x}[\varphi(\theta)(x)], \frac{\partial}{\partial x}[\varphi'(\theta)(x)]\Big) \text{ for } x \in [0, \pi],\ \varphi \in \mathcal{C}_{\frac{1}{2}} \text{ and } t \geq 0,$$

where $\varphi \in \mathcal{C}$ is defined by

$$\varphi(\theta)(x) = \varphi_0(\theta, x) \text{ for } \theta \leq 0 \text{ and } x \in [0, \pi]$$

and the norm in $C_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{C}_{\frac{1}{2}}} = \sup_{\theta\in[-r,0]} \Big(\int_0^\pi \Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}} + \sup_{\theta\in[-r,0]} \Big(\int_0^\pi \Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}}.$$

Let us pose $v(t) = z(t, x)$. Then equation (5.1) takes the following abstract form

$$\begin{cases} v''(t) = Av(t) + f(t, v_t, v_t') \text{ for } t \geq 0 \\ v_0 = \varphi \\ v_0' = \varphi'. \end{cases} \tag{5.2}$$

From [10], for all $y \in X_{\frac{1}{2}}$, $y$ is absolutely continuous and $|y|_{\frac{1}{2}} = |y'|_{L^2}$. Let $\varphi, \psi \in C^1([-r, 0]; X_{\frac{1}{2}})$, since $|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \dfrac{1}{2}(|x_1 - x_2| + |y_1 - y_2|)$, then we have

$$\begin{aligned} |f(t, \varphi, \varphi') - f(t, \psi, \psi')|_{L^2} &= \Big(\int_0^\pi |f(t, \varphi, \varphi') - f(t, \psi, \psi')|^2 dx\Big)^{\frac{1}{2}} \\ &= \Big(\int_0^\pi \Big(g\Big(t, \frac{\partial}{\partial x}[\varphi(\theta)(x)], \frac{\partial}{\partial x}[\varphi'(\theta)(x)]\Big) - g\Big(t, \frac{\partial}{\partial x}[\psi(\theta)(x)], \frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big)\Big)^2 dx\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\Big(\int_0^\pi \Big(\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)]\Big| + \Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|\Big)^2 dx\Big)^{\frac{1}{2}} \end{aligned}$$

By using the Minkowski's inequality, we have

$$\begin{aligned}|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2} &\leq \frac{1}{2}\Big(\int_0^\pi \Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\ &+\frac{1}{2}\Big(\int_0^\pi \Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\sup_{\theta\in[-r,0]}\Big(\int_0^\pi \Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\ &+\frac{1}{2}\sup_{\theta\in[-r,0]}\Big(\int_0^\pi \Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}},\end{aligned}$$

which implies that

$$|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2} \leq \frac{1}{2}\|\varphi-\psi\|_{\mathcal{C}_{\frac{1}{2}}}$$

Consequently the function $f$ satifies the condition of Corrolary 3.7. Then equation (5.2) has a unique mild solution which is defined for $t\geq 0$. For the regularity, we make the following assumptions.

($\mathbf{H}_5$) $g\in C^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R};\mathbb{R})$, such that $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are locally lipschitz continuous.

($\mathbf{H}_6$)

$$\begin{cases}\varphi\in C^3([-r,0]\times[0,\pi]) \text{ such that } \varphi(0),\varphi''(0)\in D(A),\ \varphi'(0),\varphi^{(3)}(0)\in E\\ \dfrac{\partial^2}{\partial\theta^2}\varphi(0,x)=\dfrac{\partial^2}{\partial x^2}\varphi(0,x)+\displaystyle\int_{-r}^0 g(\varphi(\theta,x))d\theta \text{ for } x\in[0,\pi],\\ \dfrac{\partial^3}{\partial\theta^3}\varphi(0)(x)=\dfrac{\partial^2}{\partial x^2}\varphi'(0,x) \text{ for } x\in[0,\pi].\end{cases}$$

**Proposition 5.1.** *Under the above assumptions, equation* (5.1) *has a unique strict solution* $u$ *defined for* $t\geq 0$ *and* $x\in[0,\pi]$.

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