Journal de Mathématiques Pures et Appliquées de Ouagadougou Volume 4 Numéro 1 (2025)

ISSN: 2756-732X **URL:https//:www.journal.uts.bf/index.php/jmpao**

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Stability of the free equilibrium state of a nonlinear age structured model for a two-sex population

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Abstract : This paper focuses on the study of the stability of the free equilibrium state of a nonlinear age structured model. The spectral properties of semigroup, the Riesz-Fréchet-Kolmogorov (RFK) criterion in L^2 combined to the Calkin algebra theory are used to obtain the stability result of the free equilibrium state.

Keywords : free equilibrium, stability, Riesz-Fréchet-Kolmogorov criterion, Calkin algebra.
2010 Mathematics Subject Classification :
(Received: 23/10/2025) (Accepted 24 /11/ 2025)

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1 Introduction

Understanding the stability of steady states of structured population is an important theme in mathematical biology, with applications from ecology to epidemiology and demography. Among the various modeling frameworks, Sharpe and Lotka in [10] and McKendrick in [5] introduced the structuring individuals by a continuous age variable leadind to the formulation of a linear PDE of transport type. This single PDE model has been extensively studied by Gurtin and MacCamy [3], Webb [13], Metz and Diekmann [6], Thieme [11], Perthame [9] and Magal [4]. In our case, the fertility rate β of the female depend on the total population of the fertile males inducing a nonlinear part in the PDE problem as described as follows:

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$$\begin{cases} \dfrac{\partial m(a,t)}{\partial t} + \dfrac{\partial m(a,t)}{\partial a} = -\mu_m(a)m(a,t) & \text{in } Q, \\ \dfrac{\partial f(a,t)}{\partial t} + \dfrac{\partial f(a,t)}{\partial a} = -\mu_f(a)f(a,t) & \text{in } Q, \\ m(a,0) = m_0(a) \text{ and } f(a,0) = f_0(a) & \text{in } Q_A, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,M)f(a,t)da & \text{in } Q_T, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,M)f(a,t)da & \text{in } Q_T, \\ M = \displaystyle\int_0^A \lambda(a)m(a,t)da & \text{in } Q_T, \end{cases} \tag{1.1}$$

where T is a positive number. We use the following notations in this work

$$Q = (0,A) \times (0,T),\ Q_A = (0,A),\ Q_T = (0,T);$$

with

$$0 \leq a_1 < a_2 \leq A \text{ and } 0 \leq b_1 < b_2 \leq A.$$

We denote the density of males and females of age a at time t respectively by $m(a,t)$ and $f(a,t)$. Moreover, μ_m and μ_f denote respectively the natural mortality rate of males and females.

We have denoted by β the positive function describing the fertility rate of female individuals that depends on a and also on

$$M = \int_0^A \lambda(a)m(a,t)da,$$

where λ is the fertility function of the male individuals. Thus the densities of newborn male and female individuals at time t are given respectively by $m(0,t) = (1-\gamma)N(t)$ and $f(0,t) = \gamma N(t)$ where

$$N(t) = \int_0^A \beta(a,M)f(a,t)da.$$

We assume that the fertility rate $\beta,\ \lambda$ and the mortality rate $\mu_f,\ \mu_m$ satisfy the demographic properties :

$$(H_1)\begin{cases} \mu_m(a) > 0, \quad \mu_f(a) > 0 \text{ a.e } a \in (0,A), \\ \\ \mu_m \in L^1_{loc}(0,A), \quad \mu_f \in L^1_{loc}(0,A), \\ \\ \displaystyle\int_0^A \mu_m(a)da = +\infty, \quad \int_0^A \mu_f(a)da = +\infty. \end{cases}$$

$$(H_2)\begin{cases} \beta(a,p) \in C([0,A]\times\mathbb{R}), \\ \\ \text{there exists a constant } \alpha_+ > 0 \text{ such that } 0 \le \beta \le \alpha_+,\ \forall (a,p) \in (0,A)\times\mathbb{R}. \end{cases}$$

We further assume that the birth function β and the fertility function λ verify the following hypothesis:

$$(H_3)\begin{cases} \beta(a,p) = \beta_1(a)\beta_2(p) \text{ for all } (a,p) \in (0,A)\times\mathbb{R}, \\ \\ \text{there exists } C > 0 \text{ such that } |\beta_2(p) - \beta_2(q)| \le C|p-q| \text{ for all } p,q \in \mathbb{R}, \\ \\ \beta_1\mu_f \in L^1(0,A). \end{cases}$$

$$(H_4)\begin{cases} \lambda \in C^1([0,A]), \\ \\ \lambda(a) \ge 0 \text{ for every } a \in [0,A], \\ \\ \lambda\mu_m \in L^1(0,A). \end{cases}$$

Under the assumptions below, the well-posedness of the system (1.1) is ensured. For more details on this result see [12].

In this paper, we analyse the free steady point of a nonlinear coupled system describing the dynamics of two-sex structured population. The article is structured as follows: the section 2 concerns the free equilibrium state and the last section is related to the the stability analysis of the free equilibrium of the problem (1.1). To achieve that goal, we use the spectral theory for the differential operator of the PDE problem, and some compactness properties of the nonlinear part of the problem.

2 Free equilibrium state

The point (m^*, f^*) is an equilibrium of (1.1) if it is a solution of the following system:

$$\begin{cases} (m^*)'(a) = -\mu_m(a)m^*(a) & \text{in } (0, A), \\ (f^*)'(a) = -\mu_f(a)f^*(a) & \text{in } (0, A), \\ m^*(0) = (1-\gamma)\displaystyle\int_0^A \beta(a, M^*)f^*(a)da, \\ f^*(0) = \gamma \displaystyle\int_0^A \beta(a, M^*)f^*(a)da, \\ M^* = \displaystyle\int_0^A \lambda(a)m^*(a)da. \end{cases} \tag{2.1}$$

Let us consider the following threshold:

$$R(M^*) = \gamma \int_0^A \beta(a, M^*)e^{-\int_0^a \mu_f(s)ds}da \text{ and } R_0 = R(0) = \gamma \int_0^A \beta(a, 0)e^{-\int_0^a \mu_f(s)ds}da \tag{2.2}$$

One can easily see that $E_0 = (0; 0)$ is solution of the problem (2.1). Thus, we study the stability of the equlibrium E_0 in the following section.

3 Stability results

In all that follows, consider the Banach space

$$\mathcal{X} = \left(L^2(0, A)\right)^2$$

with the product norm. We consider the following differential operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \longrightarrow \mathcal{X}$,

$$D(\mathcal{A}) = \left\{(m, f) \in \mathcal{X},\ m(0) = (1-\gamma)\int_0^A \beta(a, M)f(a)da \text{ and } f(0) = \gamma \int_0^A \beta(a, M)f(a)da\right\},$$

$$\mathcal{A} = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix},$$

where

$$\mathcal{D}_1 m = -\frac{dm}{da} - \mu_m m \text{ and } \mathcal{D}_2 f = -\frac{df}{da} - \mu_f f.$$

So, the problem (1.1) is equivalent to the following Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} m(t) \\ f(t) \end{pmatrix} = \mathcal{A}\begin{pmatrix} m(t) \\ f(t) \end{pmatrix}, \\ \big(m(0), f(0)\big) = \big(m_0(\cdot), f_0(\cdot)\big) \in \mathcal{X}. \end{cases} \tag{3.1}$$

3.1 Spectral properties

Before proceeding, we introduce some notations and recall some definitions of spectral theory. We consider the differential operator $A : D(A) \subset X \longrightarrow X$ and the semigroup $\{T_A(t)\}_{t\geq 0}$.

Definition 3.1.

(i) The resolvent set $\rho(A)$ of A is the set of $z \in \mathbb{C}$ such that $(A - zId_X)$ is invertible and the spectrum $\sigma(A)$ is the complementary set of $\rho(A)$ in $\mathbb{C}$.

(ii) The spectral bound $s(A)$ is defined by $s(A) = \sup\{Re\lambda,\ \lambda \in \sigma(A)\}$.

Denoting $\mathcal{L}(X)$ the set of bounded linear operators on X and $\mathcal{K}(X)$ the subset of compact operators on X. Then, we have the following definition.

Definition 3.2.

(i) We define the essential norm $||L||_{ess}$ of $L \in \mathcal{L}(X)$ by:

$$||L||_{ess} = \inf_{K\in\mathcal{K}(X)} ||L - K||_X.$$

(ii) The growth bound $\omega_0(A) \in [-\infty, \infty)$ of A is defined by:

$$\omega_0(A) = \lim_{t\to\infty} \frac{1}{t} \ln\Big(||T_A(t)||_X\Big)$$

and the essential growth bound $\omega_{ess}(A) \in [-\infty, \infty)$ by:

$$\omega_{ess}(A) = \lim_{t\to\infty} \frac{1}{t} \ln\Big(||T_A(t)||_{ess}\Big).$$

We recall that the quotient $\mathcal{L}(X)/\mathcal{K}(X)$ is called the Calkin algebra which, when providing the norm $||\hat{L}|| = ||L||_{ess}$, where $\hat{L} = L + \mathcal{K}(X)$ is a Banach algebra with unit (see [2] and references cited in for details).
The below result gives a characterization of the growth bound using the spectrum of A.

Theorem 3.3. *[2]*
The growth bound of A satisfies

$$\omega_0(A) = \max\{\omega_{ess}(A),\ s(A)\},$$

and for every $\omega > \omega_{ess}(A)$, the set $\sigma_\omega = \{\lambda \in \sigma(A),\ Re(\lambda) > \omega\}$ is finite and composed of finite algebraic multiplicity elements.

Remark 3.4.
Due to the quotient defined previously, the use of the Calkin algebra shows well why the compact operators do not affect the growth bound values. More specifically, one gets $\omega_{ess}(A+K) = \omega_{ess}(A)$ for every $K \in \mathcal{K}(X)$.

The following theorem (see [8] and references cited in) gives some conditions of stability of an equilibrium.

Theorem 3.5.
Consider E an equilibrium of problem (3.1). Then, the following assertions hold:

(i) if $\omega_0(A) < 0$, then E is locally exponentially asymptotically stable for problem (3.1);

(ii) if $\omega_0(A) > 0$ and $\omega_{ess}(A) \leq 0$, then E is unstable for problem (3.1).

3.2 Stability of the equilibrium $\mathbf{E_0}$

In the context of problem (1.1), the following result holds.

Theorem 3.6.
We have $\omega_{ess}(A) < 0$.

Proof
For the proof of the result above, we apply the technic used in the proof of Theorem 3.3 (see [8]). The concrete expression of the semigroup generated by $\mathcal{A}$ is given by:

$$T_{\mathcal{A}}(t)\begin{pmatrix} m_0 \\ \\ f_0 \end{pmatrix} = \begin{pmatrix} m_0(a-t)e^{-\int_{a-t}^{a}\mu_m(s)ds}\mathbb{I}_{\{a\geq t\}} + \varphi(t-a)e^{-\int_{a-t}^{a}\mu_m(s)ds}\mathbb{I}_{\{a<t\}} \\ \\ f_0(a-t)e^{-\int_{a-t}^{a}\mu_f(s)ds}\mathbb{I}_{\{a\geq t\}} + \psi(t-a)e^{-\int_{a-t}^{a}\mu_f(s)ds}\mathbb{I}_{\{a<t\}} \end{pmatrix},$$

where $\varphi(t) = m(0,t)$ and $\psi(t) = f(0,t)$. We decompose the operator $T_{\mathcal{A}}$ in

$$T_{\mathcal{A}}(t)\begin{pmatrix} m_0 \\ \\ f_0 \end{pmatrix}(a) = T_1(t)\begin{pmatrix} m_0 \\ \\ f_0 \end{pmatrix}(a) + T_2(t)\begin{pmatrix} m_0 \\ \\ f_0 \end{pmatrix}(a),$$

with

$$T_1(t)\begin{pmatrix} m_0 \\ \\ f_0 \end{pmatrix}(a) = \begin{cases} \left(m_0(a-t)e^{-\int_{a-t}^{a}\mu_m(s)ds}, f_0(a-t)e^{-\int_{a-t}^{a}\mu_f(s)ds}\right) \text{ if } a \geq t, \\ \\ (0,0) \text{ if } a < t. \end{cases}$$

$$T_2(t)\begin{pmatrix} m_0 \\ \\ f_0 \end{pmatrix}(a) = \begin{cases} (0,0) \text{ if } a \geq t, \\ \\ \left(\varphi(t-a)e^{-\int_{a-t}^{a}\mu_m(s)ds}, \psi(t-a)e^{-\int_{a-t}^{a}\mu_f(s)ds}\right) \text{ if } a < t. \end{cases}$$

For the operator T_1, we get the following upper bound:

$$\left\|T_1(t)\begin{pmatrix} m_0 \\ f_0 \end{pmatrix}\right\|_{\mathcal{X}} = \left(\int_t^A m_0^2(a-t)e^{-\int_{a-t}^{a}\mu_m(s)ds}da\right)^{\frac{1}{2}} + \left(\int_t^A f_0^2(a-t)e^{-\int_{a-t}^{a}\mu_f(s)ds}da\right)^{\frac{1}{2}}. \tag{3.2}$$

Using the first condition of (H_1), it follows that there exists a constant $\mu_0 > 0$ such that

$$\mu_m(a) > \mu_0 \text{ and } \mu_f(a) > \mu_0 \quad \text{a.e} \quad a \in (0, A).$$

From (3.2) we obtain

$$\left\|T_1(t)\begin{pmatrix} m_0 \\ f_0 \end{pmatrix}\right\|_{\mathcal{X}} \leq e^{-\mu_0 t}\left(\int_0^A m_0^2(u)du\right)^{\frac{1}{2}} + e^{-\mu_0 t}\left(\int_0^A f_0^2(u)du\right)^{\frac{1}{2}} = e^{-\mu_0 t}\left\|\begin{pmatrix} m_0 \\ f_0 \end{pmatrix}\right\|_{\mathcal{X}}$$

and conseqently, we get

$$\left|\left|T_1(t)\right|\right|_{\mathcal{X}} \le e^{-\mu_0 t}. \tag{3.3}$$

Remark that

$$\begin{aligned}\varphi(t) &= (1-\gamma)\int_0^t \beta(u,M)\varphi(t-u)e^{-\int_0^u \mu_m(s)ds}du + (1-\gamma)\int_t^A \beta(u,M)m_0(u-t)e^{-\int_{u-t}^u \mu_m(s)ds}du \\ &= (1-\gamma)\int_0^t \beta(u,M)\varphi(t-u)e^{-\int_0^u \mu_m(s)ds}du + (1-\gamma)\int_0^A \beta(u+t,M)m_0(u)e^{-\int_u^{u+t} \mu_m(s)ds}du\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\psi(t) &= \gamma\int_0^t \beta(u,M)\psi(t-u)e^{-\int_0^u \mu_f(s)ds}du + \gamma\int_t^A \beta(u,M)f_0(u-t)e^{-\int_{u-t}^u \mu_f(s)ds}du \\ &= \gamma\int_0^t \beta(u,M)\psi(t-u)e^{-\int_0^u \mu_f(s)ds}du + \gamma\int_0^A \beta(u+t,M)f_0(u)e^{-\int_u^{u+t} \mu_f(s)ds}du.\end{aligned} \tag{3.5}$$

Let us define

$$S_1,\ S_2\ :\ L^2(0,t) \longrightarrow L^2(0,t)$$

and

$$\hat{S}_1,\ \hat{S}_2\ :\ L^2(0,A) \longrightarrow L^2(0,t)$$

by:

$$\begin{aligned}S_1(\phi)(\xi) &= \int_0^\xi \phi(y)\beta(\xi-y,M)e^{-\int_0^{\xi-y} \mu_m(s)ds}dy,\\ S_2(\phi)(\xi) &= \int_0^\xi \phi(y)\beta(\xi-y,M)e^{-\int_0^{\xi-y} \mu_f(s)ds}dy,\\ \hat{S}_1(\phi)(\xi) &= \int_0^A \phi(y)\beta(\xi+y,M)e^{-\int_y^{\xi+y} \mu_m(s)ds}dy,\\ \hat{S}_2(\phi)(\xi) &= \int_0^A \phi(y)\beta(\xi+y,M)e^{-\int_y^{\xi+y} \mu_f(s)ds}dy.\end{aligned}$$

Thus, we get the following expression for the operator T_2:

$$T_2(t)\begin{pmatrix} m_0 \\ f_0 \end{pmatrix}(a) = \begin{cases} (0,0) & \text{if } a \ge t,\\ \left(\left((I-S_1)^{-1}\hat{S}_1 m_0(t-a)\right)e^{-\int_0^a \mu_m(s)ds}, \left((I-S_2)^{-1}\hat{S}_2 f_0(t-a)\right)e^{-\int_0^a \mu_f(s)ds}\right) & \text{if } a < t.\end{cases}$$

This latter equality is well defined. Indeed, as proved in [1], S_1 is a Volterra operator, then for all $\lambda \in \mathbb{C}\setminus\{0\}$ and $\phi \in L^2(0,t)$ fixed, we have a unique function $\varphi \in L^2(0,t)$ such that

$$(\lambda I - S_1)(\phi) = \varphi.$$

Thus $(I-S_1)^{-1}$ is well defined from $L^2(0,t)$ to $L^2(0,t)$. Since $m_0 \in L^2(0,A)$, then

$$(I-S_1)^{-1}\hat{S}_1(m_0) \in L^2(0,t).$$

In the same way, we prove that the second component of T_2 is well defined.
Remark that S_1 and S_2 are bounded, so as $(I-S_1)$ and $(I-S_2)$. Since $(I-S_1)^{-1}$ and $(I-S_2)^{-1}$ are well defined; $(I-S_1)$ and $(I-S_2)$ are bijective from $L^2(0,t)$ in itself, which is a Banach space, then $(I-S_1)^{-1}$ and $(I-S_2)^{-1}$ are bounded.
Let us define the operators $\bar{S}_1,\ \bar{S}_2\ :\ L^2(0,A) \longrightarrow L^2(0,t)$ by:

$$\begin{aligned}\bar{S}_1\phi(\xi) &= \int_0^A \phi(y)ce^{-\int_y^{\xi+y}\mu_m(s)ds}dy,\\ \bar{S}_2\phi(\xi) &= \int_0^A \phi(y)ce^{-\int_y^{\xi+y}\mu_f(s)ds}dy.\end{aligned}$$

Here, c is a positive constant. To prove the compactness of $\bar{S}_1$ and $\bar{S}_2$ for every $c>0$, we use the Riesz-Fréchet-Kolmogorov (RFK) criterion in L^2 (see for instance [8, 14]). Setting $h>0$ in $(0,A)$, taking $\mathcal{B}$ a bounded subset of $L^2(0,A)$ and denoting by $\tau_h(\phi)=\phi(\cdot+h)$ the translation operator in L^2, we have for $\phi \in \mathcal{B}$:

$$\begin{aligned}\big|\big|\tau_h(\bar{S}_1\phi)-\bar{S}_1\phi\big|\big|^2_{L^2(0,t)} &= \int_0^t\bigg(\int_0^A c\phi(y)\Big(e^{-\int_y^{\xi+y}\mu_m(s)ds}-e^{-\int_y^{\xi+y+h}\mu_m(s)ds}\Big)dy\bigg)^2 d\xi\\ &= c^2\int_0^t\bigg(\int_0^A \phi(y)e^{-\int_y^{\xi+y}\mu_m(s)ds}\Big(1-e^{-\int_{\xi+y}^{\xi+y+h}\mu_m(s)ds}\Big)dy\bigg)^2 d\xi.\end{aligned}$$

Using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\big|\big|\tau_h(\bar{S}_1\phi)-\bar{S}_1\phi\big|\big|^2_{L^2(0,t)} &\le c^2\int_0^A\phi^2(y)dy\int_0^t\int_0^A\Big(1-e^{-\int_{\xi+y}^{\xi+y+h}\mu_m(s)ds}\Big)^2dyd\xi\\ &\le c^2\int_0^A\phi^2(y)dy\int_0^t\int_0^A\Big(1-e^{-h||\mu_m||_{L^1_{loc}(0,A)}}\Big)^2dyd\xi\\ &\le c^2tA\Big(1-e^{-h||\mu_m||_{L^1_{loc}(0,A)}}\Big)^2\int_0^A\phi^2(y)dy,\end{aligned}$$

which converges to 0 uniformly on $\mathcal{B}$ when h tends to 0 since $\mathcal{B}$ is bounded. Therefore $\bar{S}_1$ is compact. Remark that for $c=\alpha_+$, we have $\hat{S}_1\phi(x) \le \bar{S}_1\phi(x)$ for all $\phi \in L^2(0,A)$ and $x \in [0,t]$. Then, $\hat{S}_1$ is also compact. Similarly, we prove that $\hat{S}_2$ is compact and so is the operator T_2. Finally since T_2 is compact,

$$||T_{\mathcal{A}}(t)||_{ess} = ||T_1(t)+T_2(t)||_{ess} = ||T_1(t)||_{ess} \le ||T_1(t)||_{\mathcal{X}}.$$

Consequently to (3.3), we get $\omega_{ess}(\mathcal{A}) \le -\mu_0$.
The linearized system to study is $u'(t)=\mathcal{A}u(t)$. Using Theorem 3.3 and since $\omega_{ess}(\mathcal{A})<0$, we just

need to study eigenvalues of $\mathcal{A}$. We thus try to solve the following system:

$$\begin{cases} \dfrac{\partial m(a,t)}{\partial t} = -\dfrac{\partial m(a,t)}{\partial a} - \mu_m(a)m(a,t) & \text{in } Q, \\ \dfrac{\partial f(a,t)}{\partial t} = -\dfrac{\partial f(a,t)}{\partial a} - \mu_f(a)f(a,t) & \text{in } Q, \\ m(0,t) = (1-\gamma)\displaystyle\int_0^A \beta(a,0)f(a,t)da & \text{in } Q_T, \\ f(0,t) = \gamma \displaystyle\int_0^A \beta(a,0)f(a,t)da & \text{in } Q_T. \end{cases} \tag{3.6}$$

As in [1], we are looking for solutions of the form $m(a,t) = \bar{m}(a)e^{\delta t}$ and $f(a,t) = \bar{f}(a)e^{\delta t}$, $\delta \in \mathbb{C}$. Thus, after replacing the latter expressions in (3.6), we obtain:

$$\begin{cases} \dfrac{d\bar{m}(a)}{da} = -(\delta + \mu_m(a))\bar{m}(a) & \text{in } (0,A), \\ \dfrac{d\bar{f}(a)}{da} = -(\delta + \mu_f(a))\bar{f}(a) & \text{in } (0,A), \\ \bar{m}(0) = (1-\gamma)\displaystyle\int_0^A \beta(a,0)\bar{f}(a)da, \\ \bar{f}(0) = \gamma \displaystyle\int_0^A \beta(a,0)\bar{f}(a)da. \end{cases} \tag{3.7}$$

Then, resolving the system (3.7), we get:

$$\begin{cases} \bar{m}(a) = \bar{m}(0)e^{-\int_0^a(\delta+\mu_m(s))ds}, \\ \bar{f}(a) = \bar{f}(0)e^{-\int_0^a(\delta+\mu_f(s))ds}, \\ \bar{m}(0) = (1-\gamma)\displaystyle\int_0^A \beta(a,0)\bar{f}(a)da, \\ \bar{f}(0) = \gamma \displaystyle\int_0^A \beta(a,0)\bar{f}(a)da. \end{cases} \tag{3.8}$$

Using the second and the last equations, we obtain the following characteristic equation:

$$\gamma \int_0^A \beta(a,0)e^{-\int_0^a(\delta+\mu_f(s))ds}da = 1.$$

Now, we can show the following theorem, where R_0 is defined as in (2.2).

Theorem 3.7.

(1) If $R_0 < 1$, then E_0 is locally exponentially asymptotically stable.

(2) If $R_0 > 1$, then E_0 is unstable.

Proof
(*1*) Suppose that $R_0 < 1$. By using $\delta = Re(\delta) + Im(\delta)i$, the characteristic equation becomes:

$$\gamma \int_0^A \beta(a,0)e^{-Re(\delta)a}\cos\big(-Im(\delta)a\big)e^{-\int_0^a \mu_f(s)ds}da + i\gamma \int_0^A \beta(a,0)e^{-Re(\delta)a}\sin\big(-Im(\delta)a\big)e^{-\int_0^a \mu_f(s)ds}da = 1$$

leading to

$$\gamma \int_0^A \beta(a,0)e^{-Re(\delta)a}\cos\big(-Im(\delta)a\big)e^{-\int_0^a \mu_f(s)ds}da = 1.$$

Consequently, if $Re(\delta) \geq 0$ we have

$$1 = \gamma \int_0^A \beta(a,0)e^{-Re(\delta)a}\cos\big(-Im(\delta)a\big)e^{-\int_0^a \mu_f(s)ds}da \leq \gamma \int_0^A \beta(a,0)e^{-\int_0^a \mu_f(s)ds}da = R_0.$$

That is absurd. So, $Re(\delta) < 0$ and $\omega_0(\mathcal{A}) = \max\{\omega_{ess}(\mathcal{A}), s(\mathcal{A})\} < 0$. From Theorem 3.5, E_0 is locally exponentially asymptotically stable.
(*2*) Suppose that $R_0 > 1$. We consider the function h defined by

$$h \ : \ \delta \longmapsto \gamma \int_0^A \beta(a,0)e^{-\delta a}e^{-\int_0^a \mu_f(s)ds}da.$$

Remark that h is strictly decreasing, with $h(0) = R_0 > 1$. Consequently, there exists $\delta > 0$ such that $h(\delta) = 1$. Thus $\omega_0(\mathcal{A}) > 0$ and since $\omega_{ess}(\mathcal{A}) \leq 0$, Theorem 3.5 implies that E_0 is unstable.

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