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SOME RESULTS IN MALCEV COLOR ALGEBRAS

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Abstract : In this paper we use the $\varepsilon-$ Jacobian to give some results in Malcev color algebras.

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1 Introduction

Malcev algebras were introduced by Malcev [8], who called these objects Moufang-Lie algebras. An anticommutative algebra is called a Malcev algebra if it satisfies the following identity

$$J(x, y, z)x = J(x, y, xz)$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of x, y and z. Since for a Lie algebra the Jacobian of any three elements vanishes, Lie algebras fall into the variety of Malcev algebras. The reader is refered to [4], [9], [12] for discussions about the relationships between Malcev algebras, exceptional Lie algebras, and physics.
Malcev superalgebras were first introduced by Okubo [11]. They generalize the class of Malcev algebras and play an important role in the geometry of smooth loops.
The concept of Malcev color algebras generalizing that of Malcev algebras and Malcev superalgebras was introduced by Daniel De la Conception in [3].
The purpose of this paper is to give some theorems in Malcev color algebras.
The paper is organized as follows. In the Section 2 the $\varepsilon - Jacobian$ in Malcev color algebras is defined and some properties are established. In Section 3 we give the main results of the paper in

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Malcev color algebras.
Throughout this paper all vector spaces and algebras considered are assumed to be finite dimensional over a fixed ground field $\mathbb{K}$ of charasteristic not 2 and G is an abelian group.

2 $\varepsilon-$ Jacobian

Definition 2.1. *(1) A $\mathbb{K}$-vector space V is said to be G-graded whenever we are given a family $(V_g)_{g\in G}$ of subspaces of V such that $V = \oplus_{g\in G} V_g$ (direct sum).*

(2) An element v of $V = \oplus_{g\in G} V_g$ is said to be homogeneous of degree $g \in G$ if $v \in V_g$.

Definition 2.2. *An algebra $(M, +, ., \times)$ is called a G-graded algebra if :*

(1) M is a G-graded vector space $M = \oplus_{g\in G} M_g$,

(2) $M_g M_h \subseteq M_{g+h}$ for all g and h in G.

Definition 2.3. *A mapping $\varepsilon : G \times G \to \mathbb{K}^*$ is called a bicharacter on G if the following identities hold for all i, j, k in G :*

(1) $\varepsilon(i, j + k) = \varepsilon(i, j)\varepsilon(i, k)$

(2) $\varepsilon(i + j, k) = \varepsilon(i, k)\varepsilon(j, k)$

(3) $\varepsilon(i, j)\varepsilon(j, i) = 1$.

We assume throughout this paper that ε is a fixed bicharacter on G. We write $\bar{x}$ for the degree of x.

Definition 2.4. *[3] A G-graded algebra M is a Malcev color algebra if for all elements x, y, z, t of M :*

(1) $xy = -\varepsilon(\bar{x}, \bar{y})yx$

(2) $$\varepsilon(\bar{y}, \bar{z})(xz)(yt) = ((xy)z)\,t + \varepsilon(\bar{x}, \bar{y} + \bar{z} + \bar{t})\,((yz)t)\,x + \varepsilon(\bar{x} + \bar{y}, \bar{z} + \bar{t})\,((zt)x)\,y + \varepsilon(\bar{x} + \bar{y} + \bar{z}, \bar{t})\,((tx)y)\,z.$$

Example 2.5. *(1) If the mapping ε is defined by $\varepsilon(i, j) = 1$ for all i, $j \in G$, then a Malcev color algebra is a G-graded Malcev algebra.*

(2) If $G = \mathbb{Z}_2$ (the additive group of integers modulo 2) and if one chooses ε such that $\varepsilon(i, j) = (-1)^{ij}$ for all i, $j \in \mathbb{Z}_2$, then Malcev color algebras are just Malcev superalgebras.

We define the ε-Jacobian J: $M \times M \times M \longrightarrow M$ by

$$J(x, y, z) = (xy)z - x(yz) - \varepsilon(\bar{y}, \bar{z})(xz)y$$

for all elements x, y, z of M.

Proposition 2.6. *Let M be a Malcev color algebra. Then for all $x, y, z \in M$, we have*

$$J(x, y, z) = (xy)z - x(yz) + \varepsilon(\bar{x}, \bar{y})y(xz).$$

Proof. Let x, y, z be three elements of the Malcev color algebra M. Then we have

$$\begin{aligned} J(x, y, z) &= (xy)z - x(yz) - \varepsilon(\bar{y}, \bar{z})(xz)y \\ &= (xy)z - x(yz) - \varepsilon(\bar{y}, \bar{z})[-\varepsilon(\overline{xz}, \bar{y})y(xz)] \\ &= (xy)z - x(yz) + \varepsilon(\bar{y}, \bar{z})\varepsilon(\bar{x} + \bar{z}, \bar{y})y(xz) \ (\textit{because} \quad \overline{xz} = \bar{x} + \bar{z}) \\ &= (xy)z - x(yz) + \varepsilon(\bar{y}, \bar{z})\varepsilon(\bar{x}, \bar{y})\varepsilon(\bar{z}, \bar{y})y(xz) \\ &= (xy)z - x(yz) + \varepsilon(\bar{x}, \bar{y})y(xz). \end{aligned}$$

□

Proposition 2.7. *Let M be a Malcev color algebra. Then for all $x, y, z \in M$, we have*

$$J(x, y, z) = -\varepsilon(\bar{x}, \bar{y})J(y, x, z).$$

Proof. Let x, y, z be three elements of the Malcev color algebra M. We have,

$$\begin{aligned} J(x, y, z) &= (xy)z - x(yz) - \varepsilon(\bar{y}, \bar{z})(xz)y \\ &= -\varepsilon(\bar{x}, \bar{y})[-\varepsilon(\bar{y}, \bar{x})(xy)z + \varepsilon(\bar{y}, \bar{x})x(yz) + \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{y}, \bar{z})(xz)y] \\ &= -\varepsilon(\bar{x}, \bar{y})[-\varepsilon(\bar{y}, \bar{x})\left(-\varepsilon(\bar{x}, \bar{y})\right)(yx)z - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{x}, \overline{yz})(yz)x + \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{y}, \bar{z})\left(-\varepsilon(\overline{xz}, \bar{y})y(xz)\right)] \\ &= -\varepsilon(\bar{x}, \bar{y})[(yx)z - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{x}, \bar{y} + \bar{z})(yz)x - \varepsilon(\bar{y}, \bar{x} + \bar{z})\varepsilon(\bar{x} + \bar{z}, \bar{y})y(xz)] \\ &= -\varepsilon(\bar{x}, \bar{y})[(yx)z - \varepsilon(\bar{y}, \bar{x})\varepsilon(\bar{x}, \bar{y})\varepsilon(\bar{x}, \bar{z})(yz)x - y(xz)] \\ &= -\varepsilon(\bar{x}, \bar{y})[(yx)z - \varepsilon(\bar{x}, \bar{z})(yz)x - y(xz)] \\ &= -\varepsilon(\bar{x}, \bar{y})J(y, x, z). \end{aligned}$$

□

Proposition 2.8. *Let M be a Malcev color algebra. Then for all $x, y, z \in M$, we have*

$$J(x, y, z) = -\varepsilon(\bar{y}, \bar{z})J(x, z, y).$$

Proof. Let x, y, z be three elements of the Malcev color algebra. Then we have,

$$\begin{aligned} J(x, y, z) &= (xy)z - x(yz) - \varepsilon(\bar{y}, \bar{z})(xz)y \\ &= -\varepsilon(\bar{y}, \bar{z})[-\varepsilon(\bar{z}, \bar{y})(xy)z + \varepsilon(\bar{z}, \bar{y})x(yz) + \varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{z})(xz)y] \\ &= -\varepsilon(\bar{y}, \bar{z})[-\varepsilon(\bar{z}, \bar{y})(xy)z - \varepsilon(\bar{z}, \bar{y})\varepsilon(\bar{y}, \bar{z})x(zy) + (xz)y] \\ &= -\varepsilon(\bar{y}, \bar{z})[-\varepsilon(\bar{z}, \bar{y})(xy)z - x(zy) + (xz)y] \\ &= -\varepsilon(\bar{y}, \bar{z})[(xz)y - x(zy) - \varepsilon(\bar{z}, \bar{y})(xy)z] \\ &= -\varepsilon(\bar{y}, \bar{z})J(x, z, y). \end{aligned}$$

□

Proposition 2.9. *Let M be a Malcev color algebra. Then for all $x, y, z \in M$, we have*

$$(1) \qquad J(x, y, z) = \varepsilon(\bar{x}, \bar{y} + \bar{z})J(y, z, x),$$

$$(2) \qquad J(x, y, z) = \varepsilon(\bar{x} + \bar{y}, \bar{z})J(z, x, y).$$

Proof. Let x, y, z be three elements of the Malcev color algebra M. We have
$J(x, y, z) = -\varepsilon(\bar{x}, \bar{y})J(y, x, z)$ and $J(x, y, z) = -\varepsilon(\bar{y}, \bar{z})J(x, z, y)$. Then:

$$(1)\qquad \begin{aligned} J(x,y,z) &= -\varepsilon(\bar{x},\bar{y})J(y,x,z) \\ &= -\varepsilon(\bar{x},\bar{y})[-\varepsilon(\bar{x},\bar{z})J(y,z,x)] \\ &= \varepsilon(\bar{x},\bar{y}+\bar{z})J(y,z,x), \end{aligned}$$

$$(2)\qquad \begin{aligned} J(x,y,z) &= -\varepsilon(\bar{y},\bar{z})J(x,z,y) \\ &= -\varepsilon(\bar{y},\bar{z})[-\varepsilon(\bar{x},\bar{z})J(z,x,y)] \\ &= \varepsilon(\bar{x}+\bar{y},\bar{z})J(z,x,y). \end{aligned}$$

□

3 Main results

Lemma 3.1. *Let M be a Malcev color algebra. Then for all $x,y,z,t \in M$, we have*

$$(1)\quad \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{y}+\bar{t},\bar{z})\,((zx)t)\,y = \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\,(t(xz))\,y,$$

$$(2)\quad \varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})\,((zt)x)\,y = \varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{t}+\bar{z})\,(x(tz))\,y.$$

Proof. Let x,y,z,t be four elements of the Malcev color algebra M. Then we have:

$$(1)\quad \begin{aligned} \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\,(t(xz))\,y &= \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})\,(t(xz))\,y \\ &= \varepsilon(\bar{y},\bar{z})[-\varepsilon(\bar{x},\bar{z})]\,(t(zx))\,y \\ &= \varepsilon(\bar{y},\bar{z})[-\varepsilon(\bar{x},\bar{z})][-\varepsilon(\bar{t},\overline{zx})]\,((zx)t)\,y \\ &= \varepsilon(\bar{y},\bar{z})\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{t},\bar{z}+\bar{x})\,((zx)t)\,y \\ &= \varepsilon(\bar{y}+\bar{x},\bar{z})\varepsilon(\bar{t},\bar{z}+\bar{x})\,((zx)t)\,y \\ &= \varepsilon(\bar{y}+\bar{x},\bar{z})\varepsilon(\bar{t},\bar{z})\varepsilon(\bar{t},\bar{x})\,((zx)t)\,y \\ &= \varepsilon(\bar{y}+\bar{x}+\bar{t},\bar{z})\varepsilon(\bar{t},\bar{x})\,((zx)t)\,y, \end{aligned}$$

$$(2)\quad \begin{aligned} \varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})\,((zt)x)\,y &= \varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})[-\varepsilon(\bar{z},\bar{t})]\,((tz)x)\,y \\ &= \varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})[-\varepsilon(\bar{z},\bar{t})][-\varepsilon(\overline{tz},\bar{x})]\,(x(tz))\,y \\ &= \varepsilon(\bar{t}+\bar{x}+\bar{y},\bar{z})\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{t}+\bar{z},\bar{x})\,(x(tz))\,y \\ &= \varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{z},\bar{x})\,(x(tz))\,y \\ &= \varepsilon(\bar{t}+\bar{x}+\bar{y},\bar{z})\varepsilon(\bar{z},\bar{t}+\bar{x})\varepsilon(\bar{t},\bar{x})\,(x(tz))\,y \\ &= \varepsilon(\bar{t}+\bar{x},\bar{z})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{t}+\bar{x})\varepsilon(\bar{t},\bar{x})\,(x(tz))\,y \\ &= \varepsilon(\bar{y},\bar{z})\varepsilon(\bar{t},\bar{x})\,(x(tz))\,y \\ &= \varepsilon(\bar{y},\bar{z})\varepsilon(\bar{y},\bar{t})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{t},\bar{x})\,(x(tz))\,y \\ &= \varepsilon(\bar{y},\bar{t}+\bar{z})\varepsilon(\bar{t},\bar{x}+\bar{y})\,(x(tz))\,y. \end{aligned}$$

□

Theorem 3.2. *Let M be a Malcev color algebra. For all $x,y,z,t \in M$, we have*
$\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz)+\varepsilon(\bar{x},\bar{y})J(t,y,xz) = \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})J(x,y,z)t+\varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x.$

Proof. Let x,y,z,t be four elements of the Malcev color algebra M.
By the definition of $\varepsilon-Jacobien$ we have
$J(x,y,z) = (xy)z - x(yz) - \varepsilon(\bar{y},\bar{z})(xz)y$
$\qquad\qquad = (xy)z - x(yz) + \varepsilon(\bar{x},\bar{y})y(xz)$ then
$J(x,y,tz) = (xy)(tz) - x\,(y(tz)) - \varepsilon(\bar{y},\bar{t}+\bar{z})\,(x(tz))\,y$ and

$J(t,y,xz) = (ty)(xz) - t\,(y(xz)) - \varepsilon(\bar{y},\bar{x}+\bar{z})\,(t(xz))\,y.$ Therefore
$\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz) + \varepsilon(\bar{x},\bar{y})J(t,y,xz) = \varepsilon(\bar{t},\bar{x}+\bar{y})(xy)(tz) - \varepsilon(\bar{t},\bar{x}+\bar{y})x\,(y(tz))$
$-\,\varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{t}+\bar{z})\,(x(tz))\,y + \varepsilon(\bar{x},\bar{y})(ty)(xz) - \varepsilon(\bar{x},\bar{y})t\,(y(xz)) - \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\,(t(xz))\,y.$
As M is a Malcev color algebra, we have $\varepsilon(\bar{t},\bar{x}+\bar{y})(xy)(tz) = \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{t},\bar{y})(xy)(tz)$
$= \varepsilon(\bar{t},\bar{x})\,((xt)y)\,z + \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\,((ty)z)\,x + \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{t},\bar{y}+\bar{z})\,((yz)x)\,t$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{y}+\bar{t},\bar{z})\,((zx)t)\,y.$
And $\varepsilon(\bar{x},\bar{y})(ty)(xz) = ((tx)y)\,z + \varepsilon(\bar{t},\bar{y}+\bar{x}+\bar{z})\,((xy)z)\,t + \varepsilon(\bar{t}+\bar{x},\bar{y}+\bar{z})\,((yz)t)\,x$
$+\,\varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})\,((zt)x)\,y.$
Therefore $\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz)+\varepsilon(\bar{x},\bar{y})J(t,y,xz) = \varepsilon(\bar{t},\bar{x})\,((xt)y)\,z+\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\,((ty)z)\,x$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{t},\bar{y}+\bar{z})\,((yz)x)\,t + \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{y}+\bar{t},\bar{z})\,((zx)t)\,y - \varepsilon(\bar{t},\bar{x}+\bar{y})x\,(y(tz))$
$-\,\varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{t}+\bar{z})\,(x(tz))\,y + ((tx)y)\,z + \varepsilon(\bar{t},\bar{y}+\bar{x}+\bar{z})\,((xy)z)\,t + \varepsilon(\bar{t}+\bar{x},\bar{y}+\bar{z})\,((yz)t)\,x$
$+\,\varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})\,((zt)x)\,y - \varepsilon(\bar{x},\bar{y})t\,(y(xz)) - \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\,(t(xz))\,y.$
By Lemma 3.1 we have:
$\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{y}+\bar{t},\bar{z})\,((zx)t)\,y - \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x}+\bar{z})\,(t(xz))\,y = 0$ and
$\varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{y},\bar{t}+\bar{z})\,(x(tz))\,y - \varepsilon(\bar{t}+\bar{y}+\bar{x},\bar{z})\,((zt)x)\,y = 0.$
As $\varepsilon(\bar{t},\bar{x})\,((xt)y)\,z + ((tx)y)\,z = 0$ then
$\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz) + \varepsilon(\bar{x},\bar{y})J(t,y,xz) =$
$\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\,((ty)z)\,x + \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x}+\bar{t},\bar{y}+\bar{z})\,((yz)x)\,t - \varepsilon(\bar{t},\bar{x}+\bar{y})x\,(y(tz))$
$+\,\varepsilon(\bar{t},\bar{y}+\bar{x}+\bar{z})\,((xy)z)\,t + \varepsilon(\bar{t}+\bar{x},\bar{y}+\bar{z})\,((yz)t)\,x - \varepsilon(\bar{x},\bar{y})t\,(y(xz))$
$= \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{x},\bar{y}+\bar{z})\,((ty)z)\,x + \varepsilon(\bar{t},\bar{x})\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t},\bar{y}+\bar{z})\,((yz)x)\,t - \varepsilon(\bar{t},\bar{x}+\bar{y})x\,(y(tz))$
$+\,\varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,((xy)z)\,t + \varepsilon(\bar{t}+\bar{x},\bar{y}+\bar{z})\,((yz)t)\,x - \varepsilon(\bar{x},\bar{y})t\,(y(xz))$
$= \varepsilon(\bar{x},\bar{y}+\bar{z})\,((ty)z)\,x + \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{y}+\bar{z})\,((yz)x)\,t - \varepsilon(\bar{t},\bar{x}+\bar{y})x\,(y(tz)) + \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,((xy)z)\,t$
$+\,\varepsilon(\bar{t},\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{y}+\bar{z})\,((yz)t)\,x - \varepsilon(\bar{x},\bar{y})[-\varepsilon(\bar{t},\bar{y}+\bar{x}+\bar{z})]\,(y(xz))\,t$
$= \varepsilon(\bar{x},\bar{y}+\bar{z})\,((ty)z)\,x + \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{y}+\bar{z})[-\varepsilon(\bar{y}+\bar{z},\bar{x})]\,(x(yz))\,t$
$-\varepsilon(\bar{t},\bar{x}+\bar{y})[-\varepsilon(\bar{x},\bar{y}+\bar{t}+\bar{z})]\,(y(tz))\,x + \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,((xy)z)\,t + \varepsilon(\bar{t},\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{y}+\bar{z})[-\varepsilon(\bar{y}+\bar{z},\bar{t})]\,(t(yz))\,x$
$+\,\varepsilon(\bar{t},\bar{y}+\bar{x}+\bar{z})\varepsilon(\bar{x},\bar{y})\,(y(xz))\,t$
$= \varepsilon(\bar{x},\bar{y}+\bar{z})\,((ty)z)\,x - \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,(x(yz))\,t + \varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{y}+\bar{z})\,(y(tz))\,x + \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,((xy)z)\,t$
$-\,\varepsilon(\bar{x},\bar{y}+\bar{z})\,(t(yz))\,x + \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{y})\,(y(xz))\,t$
$= \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,[((xy)z)\,t - (x(yz))\,t + \varepsilon(\bar{x},\bar{y})\,(y(xz))\,t]$
$+\,\varepsilon(\bar{x},\bar{y}+\bar{z})\,[((ty)z)\,x - (t(yz))\,x + \varepsilon(\bar{t},\bar{y})\,(y(tz))\,x]$
$= \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\,[(xy)z - x(yz) + \varepsilon(\bar{x},\bar{y})y(xz)]\,t + \varepsilon(\bar{x},\bar{y}+\bar{z})\,[(ty)z - t(yz) + \varepsilon(\bar{t},\bar{y})y(tz)]\,x$
$= \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})J(x,y,z)t + \varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x.$
Thus
$\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz) + \varepsilon(\bar{x},\bar{y})J(t,y,xz) = \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})J(x,y,z)t + \varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x$
for all $x,y,z,t \in$ M.

□

Theorem 3.3. *Let M be a Malcev color algebra. Then the following assertions are equivalent.*

1. *For all $x,y,z,t \in M$,*
   $\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz) + \varepsilon(\bar{x},\bar{y})J(t,y,xz) = \varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})J(x,y,z)t + \varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x.$
2. *For all $x,y,z,t \in M$,*
   $J(xz,t,y) + \varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})J(tz,x,y) = \varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})J(z,t,y)x + \varepsilon(\bar{x},\bar{z})\varepsilon(\bar{t},\bar{y})J(z,x,y)t.$

Proof. Let x,y,z,t be four elements of the Malcev color algebra M.
Let us suppose (1). By Proposition 2.9 we have $J(x,y,z) = \varepsilon(\bar{x}+\bar{y},\bar{z})J(z,x,y)$.
Then

$\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz)+\varepsilon(\bar{x},\bar{y})J(t,y,xz)$
$=\varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})J(x,y,z)t+\varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x$
implies
$\varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{x}+\bar{y},\overline{tz})J(tz,x,y)+\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{t}+\bar{y},\overline{xz})J(xz,t,y)$
$=\varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\varepsilon(\bar{x}+\bar{y},\bar{z})J(z,x,y)t+\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t}+\bar{y},\bar{z})J(z,t,y)x.$
It means that
$\varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{x}+\bar{y},\bar{t}+\bar{z})J(tz,x,y)+\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{t}+\bar{y},\bar{x}+\bar{z})J(xz,t,y)$
$=\varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})\varepsilon(\bar{x}+\bar{y},\bar{z})J(z,x,y)t+\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t}+\bar{y},\bar{z})J(z,t,y)x.$
In other words
$\varepsilon(\bar{t},\bar{x}+\bar{y})\varepsilon(\bar{x}+\bar{y},\bar{t})\varepsilon(\bar{x}+\bar{y},\bar{z})J(tz,x,y)+\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})J(xz,t,y)$
$=\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{y},\bar{z})J(z,x,y)t+\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t},\bar{z})\varepsilon(\bar{y},\bar{z})J(z,t,y)x$
that is
$\varepsilon(\bar{x}+\bar{y},\bar{z})J(tz,x,y)+\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{y},\bar{z})J(xz,t,y)$
$=\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{y},\bar{z})J(z,x,y)t+\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t},\bar{z})\varepsilon(\bar{y},\bar{z})J(z,t,y)x.$
Multiplying each term by $\varepsilon(\bar{x}+\bar{z},\bar{t})\varepsilon(\bar{z},\bar{y})$ knowing that $\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{x}+\bar{z},\bar{t})=1$ and $\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{y})=1$ we have
$\varepsilon(\bar{x}+\bar{z},\bar{t})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{x}+\bar{y},\bar{z})J(tz,x,y)+\varepsilon(\bar{x}+\bar{z},\bar{t})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{y},\bar{z})J(xz,t,y)$
$=\varepsilon(\bar{x}+\bar{z},\bar{t})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{t},\bar{x}+\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{y},\bar{z})J(z,x,y)t$
$+\varepsilon(\bar{x}+\bar{z},\bar{t})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t},\bar{z})\varepsilon(\bar{y},\bar{z})J(z,t,y)x.$
That means
$\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{y},\bar{z})J(tz,x,y)+J(xz,t,y)$
$=\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{z})J(z,x,y)t+\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{t},\bar{z})J(z,t,y)x.$
Therefore
$\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})J(tz,x,y)+J(xz,t,y)$
$=\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{z})J(z,x,y)t+\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})J(z,t,y)x$
which is the same with
$J(xz,t,y)+\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})J(tz,x,y)=\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})J(z,t,y)x+\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{x},\bar{z})J(z,x,y)t$
which is (2). Hence (1) implies (2).
Now let us suppose (2). As
$J(x,y,z)=\varepsilon(\bar{x}+\bar{y},\bar{z})J(z,x,y)$
then
$J(xz,t,y)+\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})J(tz,x,y)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})J(z,t,y)x+\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{t},\bar{y})J(z,x,y)t$
implies
$\varepsilon(\overline{xz}+\bar{t},\bar{y})J(y,xz,t)+\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})\varepsilon(\overline{tz}+\bar{x},\bar{y})J(y,tz,x)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\varepsilon(\bar{z}+\bar{t},\bar{y})J(y,z,t)x+\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z}+\bar{x},\bar{y})J(y,z,x)t.$
That means
$\varepsilon(\overline{xz}+\bar{t},\bar{y})\varepsilon(\bar{y}+\overline{xz},\bar{t})J(t,y,xz)$
$+\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})\varepsilon(\overline{tz}+\bar{x},\bar{y})\varepsilon(\bar{y}+\overline{tz},\bar{x})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\varepsilon(\bar{z}+\bar{t},\bar{y})\varepsilon(\bar{y}+\bar{z},\bar{t})J(t,y,z)x$
$+\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z}+\bar{x},\bar{y})\varepsilon(\bar{y}+\bar{z},\bar{x})J(x,y,z)t.$
That is
$\varepsilon(\bar{x}+\bar{z}+\bar{t},\bar{y})\varepsilon(\bar{y}+\bar{x}+\bar{z},\bar{t})J(t,y,xz)$
$+\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{t}+\bar{z}+\bar{x},\bar{y})\varepsilon(\bar{y}+\bar{t}+\bar{z},\bar{x})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{y},\bar{t})\varepsilon(\bar{z},\bar{t})J(t,y,z)x$
$+\varepsilon(\bar{x},\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{z},\bar{x})J(x,y,z)t.$

In other words
$\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{z},\bar{y})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{y},\bar{t})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{t})J(t,y,xz)$
$+\,\varepsilon(\bar{x},\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{t}+\bar{z}+\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x})\varepsilon(\bar{t}+\bar{z},\bar{x})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z}+\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,z)t.$
Which is the same with
$\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,xz)$
$+\,\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{t}+\bar{z},\bar{y})\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{x})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,z)t$
that means
$\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,xz)$
$+\,\varepsilon(\bar{z},\bar{t})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,z)t.$
In other words
$\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,xz)$
$+\,\varepsilon(\bar{z},\bar{y}+\bar{t})\varepsilon(\bar{t},\bar{y})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,z)t.$
Multiplying by $\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{y}+\bar{t},\bar{z})$ we have
$\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{y}+\bar{t},\bar{z})\varepsilon(\bar{x},\bar{y})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,xz)$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{y}+\bar{t},\bar{z})\varepsilon(\bar{z},\bar{y}+\bar{t})\varepsilon(\bar{t},\bar{y})J(x,y,tz)$
$=\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{y}+\bar{t},\bar{z})\varepsilon(\bar{x},\bar{y}+\bar{z})\varepsilon(\bar{x},\bar{t})\varepsilon(\bar{z},\bar{y}+\bar{t})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{y}+\bar{t},\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,z)t.$
That is
$\varepsilon(\bar{x},\bar{y})J(t,y,xz)$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{t},\bar{y})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{y},\bar{z})\varepsilon(\bar{t},\bar{z})\varepsilon(\bar{t},\bar{y})\varepsilon(\bar{z},\bar{y})J(x,y,z)t$
it means that
$\varepsilon(\bar{x},\bar{y})J(t,y,xz)+\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz)$
$=\varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x$
$+\,\varepsilon(\bar{t},\bar{x})\varepsilon(\bar{t},\bar{z})\varepsilon(\bar{t},\bar{y})J(x,y,z)t.$
Which is the same with
$\varepsilon(\bar{t},\bar{x}+\bar{y})J(x,y,tz)+\varepsilon(\bar{x},\bar{y})J(t,y,xz)=\varepsilon(\bar{x},\bar{y}+\bar{z})J(t,y,z)x+\varepsilon(\bar{t},\bar{x}+\bar{y}+\bar{z})J(x,y,z)t$
which is (1). Hence (2) implies (1).
Thus the assertions (1) and (2) are equivalent.

$\square$

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