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A METHOD FOR PARAMETERS IDENTIFICATION IN POPULATION DYNAMICS PROBLEM

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Abstract : The problems of pollution in population dynamics generally involve missing source terms, as well as missing initial or boundary conditions. This paper is concerned with identifying the pollution terms arising in the state equation of a population dynamics system with incomplete initial conditions. To this end, the so-called sentinel method is used. We prove the existence of such sentinels by solving a null-controllability problem with a control constraint. The key result of our work is an observability inequality of the Carleman type, adapted to the constraint.

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1 Introduction

Let Ω be an open and bounded domain of $\mathbb{R}^N$, where $N \in \{1,2,3\}$, with a boundary Γ of class C^2. For a time $T > 0$ and the life expectancy of an individual $A > 0$, define the following sets:

$$U = (0,T) \times (0,A), \quad Q = U \times \Omega, \quad Q_A = (0,A) \times \Omega, \quad Q_T = (0,T) \times \Omega,$$

$$\Sigma = U \times \Gamma, \quad \Sigma_1 = U \times \Gamma_1,$$

where Γ_1 is a nonempty open subset of Γ. We denote by ν the outer normal on Γ. We now consider the following problem:

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu y = \xi + \sum_{i=1}^{M} \lambda_i \hat{\xi}_i & \text{in } Q, \\ y(0,a,x) = y^0 + \tau \hat{y}^0 & \text{in } Q_A, \\ y(t,0,x) = \int_0^A \beta(t,a,x) y(t,a,x)\, da & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_1, \\ \frac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma \setminus \Sigma_1. \end{cases} \tag{1.1}$$

It is assumed that $\mu \geq 0$ and $\beta \geq 0$. The parameters of the problem have the following meanings: the final time $T > 0$ represents the horizon of the problem, the bound $A > 0$ represents the life expectancy, β is the natural fertility rate, and the function $\mu = \mu(t,a,x)$ is the natural death rate of individuals aged a at time $t > 0$ and position x. The functions ξ and y^0 are known, with $\xi \in L^2(Q)$ and $y^0 \in L^2(\Omega)$. However, the terms $\sum_{i=1}^{M} \lambda_i \hat{\xi}_i$ (the so-called pollution term) and $\tau \hat{y}^0$ (the so-called perturbation term) are unknown. Here, $\hat{\xi}_i$ and $\hat{y}^0$ are renormalized and represent the size of the pollution and the perturbation, respectively.

$$\|\hat{\xi}_i\|_{L^2(Q)} \leq 1 \text{ for } i = 1, \cdots, M \text{ and } \|\hat{y}^0\|_{L^2(\Omega)} \leq 1.$$

So that the real numbers $\{\lambda_i\}_{1\leq i\leq M}$ and τ are sufficiently small, and the functions $\hat{\xi}_i$, for $1 \leq i \leq M$, are linearly independent.

In the model (1.1), we are interested in identifying the parameters λ_i in the state equation, independently of the variation $\tau \hat{y}^0$ around the initial data. To identify these parameters, we use the sentinel method. In this paper, we construct sentinels when the supports of the observation function and the control function are contained in two distinct open subsets of $\mathbb{R}^N$ (see Nakoulima [14]).

The theory of sentinels relies on three features: a state equation, an observation function, and a control function w to be determined.

- **A state equation**, represented here by (1.1), for which we assume that (1.1) has a unique solution denoted by $y = y(t, a, x, \lambda, \tau) = y(\lambda, \tau)$, depending on two parameters, $\lambda = \{\lambda_1, \ldots, \lambda_M\}$ and τ, in some relevant space. We assume the following [2]:

$$(H1) \quad : \quad \begin{cases} \beta \in L^\infty_+(Q), \quad \beta(t,a,x) \geq 0 \quad \text{in} \quad Q, \\ \exists a_1 \in (0, A), \quad \beta(a, \cdot, \cdot) = 0 \quad \text{for} \quad a \in (a_1, A). \end{cases}$$

$$(H2) \quad : \quad \mu \in L^\infty_{\text{loc}}\left(]0, A[; L^\infty((0,T) \times \Omega)\right), \quad \mu \geq 0 \quad \text{a.e. in} \quad Q_A.$$

$$(H3) \quad : \quad \begin{cases} \lim\limits_{a \to A} \int_0^t \mu(a - \iota, t - \iota, x)\, d\iota = +\infty, \quad \text{a.e. in} \quad Q_A, \\ \lim\limits_{a \to A} \int_0^a \mu(a - \iota, t - \iota, x)\, d\iota = +\infty, \quad \text{a.e. in} \quad Q_A. \end{cases}$$

- **An observation** y_{obs}. Let $\mathcal{O} \subset \Omega$ be a non-empty open subset called the observation set. The observation is the value of y in $\mathcal{O}$ over the time interval $[0, T]$. We denote this observation by y_{obs}, and it is given by the equation:

$$y_{\text{obs}} = m_0 \in L^2(U \times \mathcal{O}), \tag{1.2}$$

where m_0 is the observation function.

- **A function** $S = S(\lambda, \tau)$ is called a "sentinel". Let $h_0 \in L^2(U \times \mathcal{O})$ be a given function. Let ω be another open, non-empty subset of Ω, such that $\omega \neq \mathcal{O}$. For a control function $\mathrm{w} \in L^2(U \times \omega)$, we define the functional $S(\lambda, \tau)$ as:

$$S(\lambda, \tau) = \int_U \int_{\mathcal{O}} h_0\, y(\lambda, \tau)\, dt\, da\, dx + \int_U \int_\omega \mathrm{w}\, y(\lambda, \tau)\, dt\, da\, dx. \tag{1.3}$$

We say that S defines a sentinel for the problem (1.1) if there exists a control w such that:

- S is insensitive (to first order) with respect to the missing terms $\tau \hat{y}^0$, which means:

$$\frac{\partial S}{\partial \tau}(0,0) = 0 \quad \forall\, \hat{y}^0. \tag{1.4}$$

- S is sensitive (to first order) with respect to the pollution terms $\lambda_i \hat{\xi}_i$:

$$\frac{\partial S}{\partial \lambda_i}(0,0) = c_i, \quad 1 \leq i \leq M, \tag{1.5}$$

where c_i are given constants, not all of which are identically zero.

- The control w has minimal norm in $L^2(U \times \omega)$ among the admissible controls. That is:

$$\|\mathrm{w}\|_{L^2(U \times \omega)} = \min_{u \in E} \|u\|_{L^2(U \times \omega)}, \tag{1.6}$$

where

$$E = \left\{ u \in L^2(U \times \omega) \mid (u, S(u)) \text{ satisfies } (1.3) - (1.5) \right\}.$$

Several authors have studied sentinel problems. We refer to [9], [11], and [14] for further details. In [9], G. M. Mophou and O. Nakoulima studied the problem of sentinels with given sensitivity. O. Bodart and collaborators applied the sentinel method in [23] to identify an unknown boundary. In [24], B. Ainseba and collaborators used the sentinel method to identify pollution parameters in a river. Recently, the author S. Sawadogo introduced the concept of distributed sentinels in [26] within the framework of population dynamics equations to study a population subject to migratory phenomena.

In this paper, we apply the sentinel method to identify parameters in population dynamics with age dependence, spatial structure, and incomplete data. The problem is as follows: Given $h_0 \in L^2(U \times \mathcal{O})$, find a control function $\mathrm{w} \in L^2(U \times \omega)$ such that if $y = y(\lambda, \tau)$ is the solution of equation (1.1) and S is defined by equation (1.3), then conditions (1.4) and (1.5) hold.

In the following, we assume without loss of generality that:

$$\xi = 0 \quad \text{in} \quad Q \quad \text{and} \quad y^0 = 0 \quad \text{in} \quad Q_A.$$

Remark 1.1. *Consider the function $y_\tau = \frac{\partial y}{\partial \tau}$, where y corresponds to the parameter values $\lambda = 0$ and $\tau = 0$. Similarly, define the function $y_{\lambda_i} = \frac{\partial y}{\partial \lambda_i}$, where y corresponds to the parameter values $\lambda_i = 0$ and $\tau = 0$. The functions y_τ and y_{λ_i} are the solutions of the following problems:*

$$\begin{cases} \frac{\partial y_\tau}{\partial t} + \frac{\partial y_\tau}{\partial a} - \Delta y_\tau + \mu y_\tau = 0 & \text{in} \quad Q, \\ y_\tau(0, a, x) = \hat{y}^0 & \text{in} \quad Q_A, \\ y_\tau(t, 0, x) = \int_0^A \beta(t, a, x) y_\tau(t, a, x)\, da & \text{in} \quad Q_T, \\ y_\tau = 0 & \text{on} \quad \Sigma_1, \\ \frac{\partial y_\tau}{\partial \nu} = 0 & \text{on} \quad \Sigma \setminus \Sigma_1. \end{cases} \tag{1.7}$$

$$\begin{cases} \frac{\partial y_{\lambda_i}}{\partial t} + \frac{\partial y_{\lambda_i}}{\partial a} - \Delta y_{\lambda_i} + \mu y_{\lambda_i} = \hat{\xi}_i & \text{in} \quad Q, \\ y_{\lambda_i}(0, a, x) = 0 & \text{in} \quad Q_A, \\ y_{\lambda_i}(t, 0, x) = \int_0^A \beta(t, a, x) y_{\lambda_i}(t, a, x)\, da & \text{in} \quad Q_T, \\ y_{\lambda_i} = 0 & \text{on} \quad \Sigma_1, \\ \frac{\partial y_{\lambda_i}}{\partial \nu} = 0 & \text{on} \quad \Sigma \setminus \Sigma_1. \end{cases} \tag{1.8}$$

Under the assumptions $(H_1)-(H_3)$, the linear problems (1.7) and (1.8) each have a unique solution: y_τ such that $y_\tau(t, A, x) = 0$, and y_{λ_i} such that $y_{\lambda_i}(t, A, x) = 0$. For the details of the proof, we refer to [4, 9, 17].

Remark 1.2. *If the function S defined by (1.3)-(1.5) exists, then it is unique since w verifies (1.6). In this case, to estimate the parameters λ_i, one proceeds as follows: Assume that the solution of the state equation (1.1) when $\lambda = 0$ and $\tau = 0$ is known. Then one has the following information:*

$$S(\lambda, \tau) - S(0, 0) \simeq \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0, 0).$$

Therefore, fixing $i, j \in \{1, \ldots, M\}$ and choosing i and j such that

$$\frac{\partial S}{\partial \lambda_j}(0, 0) = 0 \quad \textit{for} \quad j \neq i \quad \textit{and} \quad \frac{\partial S}{\partial \lambda_i}(0, 0) = 1,$$

one obtains the following estimate of the parameter λ_i*:*

$$\lambda_i \simeq \frac{1}{c_i}\left(S(\lambda,\tau) - S(0,0)\right).$$

Definition 1.3. *We will refer to the function* S *given by* (1.3)-(1.5) *as the sentinel function with given* $\{c_i\}$ *sensitivity.*

Let χ_ω be the characteristic function of the set ω. We set

$$Y_\lambda = \text{Span}\left\{y_{\lambda_1}\chi_\omega, \dots, y_{\lambda_M}\chi_\omega\right\}, \tag{1.9}$$

the vector subspace of $L^2(U\times\omega)$ generated by the M independent functions $y_{\lambda_i}\chi_\omega$, $1 \le i \le M$. We denote by $Y_\lambda^\perp$ the orthogonal of Y_λ in $L^2(U\times\omega)$.

$$\begin{cases} \text{any function } k \in Y_\lambda \text{ such that} \\ \frac{\partial k}{\partial t} + \frac{\partial k}{\partial a} - \triangle k + \mu k = 0, \text{in } U\times\omega, \text{ is identically zero in } U\times\omega. \end{cases} \tag{1.10}$$

Next, we consider the following general null-controllability problem: Given $h \in L^2(Q)$, find $v \in L^2(U\times\omega)$ such that

$$v \in Y_\lambda^\perp, \tag{1.11}$$

and such that $q = q(t,a,x,v) \in L^2(Q)$ which is solution of

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q &= \beta q(t,0,x) + h + v\chi_\omega & \text{in} & Q, \\ q(T,a,x) &= 0 & \text{in} & Q_A, \\ q(t,A,x) &= 0 & \text{in} & Q_T, \\ q &= 0 & \text{on} & \Sigma_1, \\ \frac{\partial q}{\partial \nu} &= 0 & \text{on} & \Sigma\setminus\Sigma_1; \end{cases} \tag{1.12}$$

satisfies

$$q(0,a,x,v) = 0 \text{ in } Q_A; \tag{1.13}$$

with v of minimal norm in $L^2(U\times\omega)$, that is

$$\|v\|_{L^2(U\times\omega)} = \min_{\overline{w}\in\varepsilon}\|\overline{w}\|_{L^2(U\times\omega)}; \tag{1.14}$$

where

$$\varepsilon = \left\{\overline{v} \in Y_\lambda^\perp \text{ such that } (\overline{v}, \overline{q} = q(t,a,x,\overline{v})) \text{ is subject to } (1.12)-(1.13)\right\} \tag{1.15}$$

For the evolution equations, other topics such as exact controllability and approximate controllability are considered. For example, in [27], the exact controllability of semi-linear stochastic evolution equations is studied, and in [28], the interior approximate controllability of the semi-linear heat equation is proved.

For the problem (1.11)–(1.14), two main aspects are considered. The first one consists of solving the null-controllability problem, and the second one consists of characterizing the optimal solution of (1.14) by some optimality system. The problem (1.11)–(1.14) is solved when $Y_\lambda = \{0\}$ (i.e., the setting without constraints or free constraints) in several cases by various methods [2, 3]. In the present paper, both aspects are considered in the general setting $Y_\lambda \neq \{0\}$. More precisely, we have the following results:

Theorem 1.4. *Assume that the above hypotheses on* Ω, ω, $\mathcal{O}$, *and the data of the equation* (1.1) *are satisfied. Then the existence of the sentinel function in* (1.3)–(1.6) *holds if and only if the null controllability problem with a constraint on the control in* (1.11)–(1.14) *holds.*

The proof of the null controllability problem with a constraint on the control in (1.11)–(1.14) relies on the existence of a function θ and a Carleman inequality adapted to the constraint (cf. Subsection 2.2), for which we have the following result:

Theorem 1.5. *Assume that the hypotheses of Theorem 1.4 and the condition* (1.10) *are satisfied. Then there exists a positive weight function* θ *such that, for any function* $h \in L^2(Q)$ *with* $\theta h \in L^2(Q)$, *the null controllability problem with a constraint on the control in* (1.11)–(1.14) *holds. Moreover, the control is given by:*

$$\hat{v}_\theta = -(\hat{\rho}_\theta - P\hat{\rho}_\theta \chi_\omega)\chi_\omega, \tag{1.16}$$

where $\hat{\rho}_\theta$ *is a solution of:*

$$\begin{cases} \frac{\partial \hat{\rho}_\theta}{\partial t} + \frac{\partial \hat{\rho}_\theta}{\partial a} - \Delta \hat{\rho}_\theta + \mu \hat{\rho}_\theta & = 0 & in & Q, \\ \hat{\rho}_\theta(t,0,x) & = \int_0^A \beta(t,a,x)\hat{\rho}_\theta(t,a,x)\,da & in & Q_T, \\ \hat{\rho}_\theta & = 0 & on & \Sigma_1, \\ \frac{\partial \hat{\rho}_\theta}{\partial \nu} & = 0 & on & \Sigma \setminus \Sigma_1. \end{cases} \tag{1.17}$$

and P *is the orthogonal projection operator from* $L^2(U \times \omega)$ *into* Y_λ.

The remainder of the paper is organized as follows. Section 2 is devoted to some preliminary results. In this section, we prove Theorem 1.4 and establish the inequality adapted to the constraint (1.11). In Section 3, we prove the existence and uniqueness of the solution for the controllability problem (1.11)–(1.14) of Theorem 1.4 and provide the proof of Theorem 1.5. Finally, Section 4 presents the expression of the sentinel S defined by (1.3)–(1.5), as well as the estimate of the parameters λ_i.

2 Preliminary results

2.1 Proof of Theorem 1.4

Since y_τ and y_λ are solutions of equations (1.7) and (1.8), respectively, the insensitivity condition (1.4) and the sensitivity conditions (1.5) hold if and only if the following hold:

$$\int_U \int_{\mathcal{O}} h_0 y_\tau \,dt\,da\,dx + \int_U \int_\omega \mathrm{w} y_\tau \,dt\,da\,dx = 0, \quad \forall \hat{y}^0, \quad \|\hat{y}^0\|_{L^2(Q_A)} \le 1, \tag{2.1}$$

and

$$\int_U \int_{\mathcal{O}} h_0 y_{\lambda_i} \,dt\,da\,dx + \int_U \int_\omega \mathrm{w} y_{\lambda_i} \,dt\,da\,dx = c_i, \quad 1 \le i \le M. \tag{2.2}$$

In order to transform equation (2.1), we introduce the classical adjoint state. More precisely, we consider the solution $q = q(t, a, x)$ of the linear problem

$$\left\{\begin{array}{rclcc} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q & = & \beta q(t,0,x) + h_0 \chi_{\mathcal{O}} + \mathrm{w}\chi_\omega & \text{in} & Q, \\ q(T,a,x) & = & 0 & \text{in} & Q_A, \\ q(t,A,x) & = & 0 & \text{in} & Q_T, \\ q & = & 0 & \text{on} & \Sigma_1, \\ \frac{\partial q}{\partial \nu} & = & 0 & \text{on} & \Sigma \setminus \Sigma_1; \end{array}\right. \tag{2.3}$$

where $\chi_{\mathcal{O}}$ and χ_ω are the indicator functions for the respective open sets $\mathcal{O}$ and ω. There is a unique solution in $L^2(Q)$ as a consequence of the fixed-point theorem for contracting mappings [3]. The so-called adjoint state q depends on the unknown function w, and its utility comes from the following process.

First, we multiply both sides of the differential equation in (2.3) by y_τ and integrate by parts over Q.

$$\int_U \int_{\mathcal{O}} h_0 y_\tau \, dt\, da\, dx + \int_U \int_\omega \mathrm{w} y_\tau \, dt\, da\, dx = \int_0^A \int_\Omega q(0,a,x)\hat{y}^0 \, da\, dx, \tag{2.4}$$
$$\forall\, \hat{y}^0 \in L^2(Q_A), \quad \|\hat{y}^0\|_{L^2(Q_A)} \leq 1.$$

Thus, the condition (1.4) (or (2.1)) holds if and only if

$$q(0,a,x) = 0, \quad \text{a.e. } (a,x) \in (0,A) \times \Omega. \tag{2.5}$$

Next, multiplying both sides of the differential equation in (2.3) by $y_{\lambda_i} \in L^2(Q)$, which is the solution of (1.8), and integrating by parts over Q, we obtain

$$\int_U \int_{\mathcal{O}} h_0 y_{\lambda_i} \, dt\, da\, dx + \int_U \int_\omega \mathrm{w} y_{\lambda_i} \, dt\, da\, dx = \int_U \int_\Omega q\hat{\xi}_i \, dt\, da\, dx, \quad 1 \leq i \leq M. \tag{2.6}$$

Thus, the condition (1.5) (or (2.2)) is equivalent to

$$\int_\Omega q\hat{\xi}_i \, dt\, da\, dx = c_i, \quad 1 \leq i \leq M. \tag{2.7}$$

Therefore, the above considerations show that the existence of the sentinel defined by (1.3)–(1.5) holds if and only if the following null controllability problem with constraints on the state q holds: Given $h_0 \in L^2(U \times \mathcal{O})$, find w of minimal norm in $L^2(U \times \omega)$ such that the pair (w, q) satisfies (2.3), (2.5), and (2.7).

Actually, condition (1.5) (or (2.7) on the state q) is equivalent to a constraint on the control. Indeed, let Y_λ be the real vector subspace of $L^2(U \times \omega)$ defined in (1.9). Since Y_λ is finite-dimensional, there exists a unique $\mathrm{w}_0 \in Y_\lambda$ such that

$$c_i - \int_U \int_{\mathcal{O}} h_0 y_{\lambda_i} dtdadx = \int_U \int_\omega \mathrm{w}_0 y_{\lambda_i} dtdadx, \quad 1 \leq i \leq M.$$

Therefore, the condition (2.2) or (2.7) holds if and only if

$$\mathrm{w} - \mathrm{w}_0 = v \in Y_\lambda. \tag{2.8}$$

Consequently, replacing w by $v + \mathrm{w}_0$ in (2.3), then setting

$$h = h_0 \chi_{\mathcal{O}} + \mathrm{w}_0 \chi_\omega \in L^2(Q), \tag{2.9}$$

we finally deduce that we have the existence of the sentinel $(1.3) - (1.5)$ if and only if, null controllability with constraint on the control $(1.11) - (1.14)$ holds ■

2.2 An adapted Carleman inequality

The observability inequality we are looking for is a consequence of Carleman's inequality. We consider an auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) > 0 \quad \forall \quad x \in \Omega, \quad \psi(x) = 0 \quad \forall \quad x \in \Gamma, \quad |\nabla \psi(x)| \neq 0 \quad \forall \quad x \in \overline{\Omega} - \omega_0, \tag{2.10}$$

where ω_0 denotes any open set such that $\overline{\omega_0} \subset \omega$ (for example, ω_0 can be some small enough open ball). Such a function ψ exists according to A. Fursikov and O. Yu. Imanuvilov [7].

We define, for any positive parameter λ, the following weight functions:

$$\varphi(t,a,x) = \frac{e^{\lambda \psi(x)}}{at(T-t)}, \quad \alpha(t,a,x) = \frac{e^{2\lambda \|\psi\|_\infty} - e^{\lambda \psi(x)}}{at(T-t)}. \tag{2.11}$$

Since φ does not vanish on Q, we set

$$\theta = \frac{e^{s\alpha}}{\varphi\sqrt{\varphi}} \quad \text{or} \quad \frac{1}{\theta} = \varphi\sqrt{\varphi} e^{-s\alpha}. \tag{2.12}$$

Remark 2.1. $\frac{1}{\theta} = \varphi\sqrt{\varphi}e^{-s\alpha}$ *is defined on* $\overline{Q} = [0;T] \times [0;A] \times \overline{\Omega}$ *by*
$\frac{1}{\theta}(t,a,x) = \begin{cases} \varphi^{\frac{3}{2}}(t,a,x)e^{-s\alpha(t,a,x)} & \text{on }]0;T[\times]0,A[\times\overline{\Omega}, \\ 0 \quad \text{on} \quad \overline{Q} - \left(]0;T[\times]0,A[\times\overline{\Omega}\right); \end{cases}$

and we have the following limits :
$\lim\limits_{(t,a,x)\to(0,0,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta(0,0,x)}$; $\lim\limits_{(t,a,x)\to(0,a,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta(0,a,x)}$;
$\lim\limits_{(t,a,x)\to(t,0,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta(t,0,x)}$; $\lim\limits_{(t,a,x)\to(T,0,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta(T,0,x)}$;
$\lim\limits_{(t,a,x)\to(0,A,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta(0,A,x)}$.
Thus $\frac{1}{\theta}$ is continuous on $\overline{Q}$ and since $\overline{Q}$ is bounded in $\mathbb{R}^{N+2}$ then $\frac{1}{\theta}$ is bounded.

We adopt the following notations :

$$\begin{cases} L = \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - \Delta + \mu I, \\ L^* = -\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial a} - \Delta + \mu I, \\ \mathcal{V} = \left\{\rho \in C^\infty\left(\overline{Q}\right), \rho_{|_{\Sigma_1}} = 0, \quad \frac{\partial \rho}{\partial \nu}_{|_{\Sigma\setminus\Sigma_1}} = 0.\right\} \end{cases} \tag{2.13}$$

Lemma 2.2. *Assume that* (1.10) *holds. Let θ be the function given by* (2.12) *and P be the operator defined as in Theorem 1.5. Then there exists a positive constant C such that for any $\rho \in \mathcal{V}$:*

$$\int_Q \frac{1}{\theta^2}|\rho|^2 dtdadx \leq C\left[\int_Q |L\rho|^2\, dtdadx + \int_0^T\int_0^A\int_\omega |\rho - P\rho|^2 dtdadx,\right]. \tag{2.14}$$

The proof of this lemma requires what we call the global Carleman's inequality.

Proposition 2.3. *(Global Carleman's inequality). Let ψ, φ and α be the functions defined respectively as in* (2.10) $-$ (2.11)*. Then, there exists $\lambda_0 > 1$ and $s_0 > 1$ and there exists $C > 0$ such that, for any $\lambda \geq \lambda_0$, for any $s \geq s_0$ and for any $\rho \in \mathcal{V}$ the following inequality holds:*

$$\begin{aligned}\int_Q \tfrac{e^{-2s\alpha}}{s\varphi}\left(|\rho_t + \rho_a|^2 + |\Delta\rho|^2\right) dtdadx + \int_Q s\lambda^2\varphi e^{-2s\alpha}|\nabla\rho|^2 dtdadx \\ + \int_Q s^3\lambda^4\varphi^3 e^{-2s\alpha}|\rho|^2 dtdadx \\ \leq C\left[\int_Q e^{-2s\alpha}\,|L\rho|^2\, dtdadx + \int_0^T\int_0^A\int_\omega s^3\lambda^4\varphi^3 e^{-2s\alpha}|\rho|^2 dtdadx\right].\end{aligned} \tag{2.15}$$

Proof. We refer to [2] and [15] ☐

According to the definition of φ and α given by (2.11), the function θ given by (2.12) is positive and $\frac{1}{\theta} = \varphi\sqrt{\varphi}e^{-s\alpha}$ is bounded. So, replacing $\frac{e^{s\alpha}}{\varphi\sqrt{\varphi}}$ by θ in (2.15) the following inequality holds:

$$\int_Q \frac{1}{\theta^2}|\rho|^2 dtdadx \leq C\left[\int_Q \frac{1}{\theta^2\varphi^3 s^3\lambda^4}\,|L\rho|^2\, dtdadx + \int_0^T\int_0^A\int_\omega \frac{1}{\theta^2}|\rho|^2 dtdadx\right]. \tag{2.16}$$

As a consequence of the boundedness of $\frac{1}{\theta}$ and $\frac{1}{\varphi^3 s^3\lambda^4}$, we get the next observability inequality:

$$\int_Q \frac{1}{\theta^2}|\rho|^2 dtdadx \leq C\left[\int_Q |L\rho|^2\, dtdadx + \int_0^T\int_0^A\int_\omega |\rho|^2 dtdadx\right]. \tag{2.17}$$

Proof. of lemma 2.2.
The proof uses a well know compactness-uniqueness argument and the inequality (2.17). Indeed suppose that (2.15) does not holds. then

$$\left\{\begin{array}{l}\forall j \in \mathbb{N}^*, \exists \rho_j \in \mathcal{V}, \displaystyle\int_Q \tfrac{1}{\theta^2}|\rho_j|^2 dtdadx = 1, \\ \displaystyle\int_Q |L\rho_j|^2\, dtdadx \leq \tfrac{1}{j} \ \text{ and} \int_U\int_\omega |\rho_j - P\rho_j|^2 dtdadx \leq \tfrac{1}{j}.\end{array}\right. \tag{2.18}$$

The forthcoming proof consists of extracting some subsequence, still denoted $(\rho_j)_j$ such that the following contradiction holds

$$\lim_{j\to+\infty}\int_Q \frac{1}{\theta^2}|\rho_j|^2 dtdadx = 0.$$

Denote by $(h \mid g)_{L^2(U\times\omega)}$ the natural scalar product in the Hilbert space $L^2(U\times\omega)$. Let $\{k_1, k_2, ..., k_M\}$ be some orthonormal basis of Y_λ.

Step1. We first show that for any $i = 1, 2, \ldots, M$, the numerical sequence $\left((\rho_j \mid k_i)_{L^2(U\times\omega)}\right)_{j\in\mathbb{N}^*}$ is bounded, or equivalently, that the sequence $\left(\|P\rho_j\|^2_{L^2(U\times\omega)}\right)_j$ is bounded.

Star with the norm inequality

$$\begin{aligned}\left(\int_U\int_\omega \tfrac{1}{\theta^2}|P\rho_j|^2 dtdadx\right)^{\frac{1}{2}} &\leq \left(\int_U\int_\omega \tfrac{1}{\theta^2}|\rho_j|^2 dtdadx\right)^{\frac{1}{2}} \\ &+ \left(\int_U\int_\omega \tfrac{1}{\theta^2}|\rho_j - P\rho_j|^2 dtdadx\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\frac{1}{\theta^2}$ is bounded and by (2.18) it follows that there is some number γ

$$\forall j \in \mathbb{N}^*, \int_U\int_\omega \frac{1}{\theta^2}|P\rho_j|^2 dtdadx \leq \gamma. \tag{2.19}$$

Since Y_λ is finite dimensional, norms are equivalent. Particularly the mappings

$$k \mapsto \int_U\int_\omega |k|^2 dtdadx \quad \text{and} \quad k \mapsto \int_U\int_\omega \frac{1}{\theta^2}|k|^2 dtdadx,$$

are equivalent norm on Y_λ. There is then some number γ'

$$\forall j \in \mathbb{N}^*, \int_U\int_\omega |P\rho_j|^2 dtdadx \leq \gamma'.$$

The relation $(\rho_j - P\rho_j) \in Y_\lambda^\perp$, $\forall j \in \mathbb{N}^*$ means the following

$$(\rho_j - P\rho_j \mid k_i)_{L^2(U\times\omega)} = 0 \quad \forall i, \quad 1 \leq i \leq M, \quad \forall j \in \mathbb{N}^*.$$

Thus

$$P\rho_j = \sum_{i=1}^{M} (P\rho_j \mid k_i)_{L^2(U\times\omega)} k_i = \sum_{i=1}^{M} (\rho_j \mid k_i)_{L^2(U\times\omega)} k_i, \tag{2.20}$$

and from orthogonality

$$\int_U\int_\omega |P\rho_j|^2\, dtdadx = \sum_{i=1}^{M} \left|(\rho_j \mid k_i)_{L^2(U\times\omega)}\right|^2 = \|P\rho_j\|^2_{L^2(U\times\omega)}. \tag{2.21}$$

Thus

$$\|P\rho_j\|^2_{L^2(U\times\omega)} \leq \gamma'. \tag{2.22}$$

Step 2. Since $(P\rho_j)_{j\in\mathbb{N}^*}$ is bounded and

$$\|\rho_j - P\rho_j\|^2_{L^2(U\times\omega)} = \int_U\int_\omega |\rho_j - P\rho_j|^2\, dtdadx \to 0,$$

then the sequence $(\rho_j)_{j\in\mathbb{N}^*}$ is bounded. There is some weakly convergence subsequence still denoted by $(\rho_j)_{j\in\mathbb{N}^*}$ such that :

$$\rho_j \rightharpoonup g \quad \text{weakly} \quad \text{in} \quad L^2(U\times\omega). \tag{2.23}$$

Since sub-sequences have the same limit as convergence sequence

$$\rho_j - P\rho_j \longrightarrow 0 \text{ strongly in } L^2(U\times\omega). \tag{2.24}$$

Next, we deduce from the compactness of P (because Y_λ is finite dimensional) that there exists $\zeta \in Y_\lambda$ such that

$$P\rho_j \longrightarrow \zeta \text{ stronly in } L^2(U\times\omega). \tag{2.25}$$

We deduce from (2.24) and (2.25) that $\rho_j \longrightarrow g = \zeta$ strongly in $L^2(U\times\omega)$. Thanks to the continuity of P, we have (2.24) and (2.25) that $P\rho_j \longrightarrow Pg$ strongly in $L^2(U\times\omega)$. Therefore, $Pg = g$ and so $g \in Y_\lambda$.

Step 3. In fact, we have $g = 0$. Indeed , from (2.18), we also have $L\rho_j \longrightarrow 0$ strongly in $L^2(Q)$. Thus $L\rho_j \longrightarrow 0$ strongly in $L^2(U\times\omega)$. We conclude that $L\rho_j \rightharpoonup 0$ weakly in $\mathcal{D}'(U\times\omega)$. and so $Lg = 0$. The assumption (1.10) implies $g = 0$ on $U\times\omega$. Finally, $\rho_j \longrightarrow 0$ strongly in $L^2(U\times\omega)$.

Step 4. Since $\rho_j \in \mathcal{V}$, it follows from the observability inequality (2.17) that

$$\int_Q \frac{1}{\theta^2}|\rho_j|^2 dtdadx \leq C\left[\int_Q |L\rho_j|^2\, dtdadx + \int_0^T\int_0^A\int_\omega |\rho_j|^2 dtdadx\right].$$

Then, the conclusions in the third step, yield that $\int_Q \frac{1}{\theta^2}|\rho_j|^2 dtdadx \longrightarrow 0$ when $j \longrightarrow +\infty$. The proof is now completed.

□

3 Null controllability with constraint on the control

The main tool used is the observability inequality (2.14), adapted to the constraint.

3.1 Existence of optimal control variable for null controllability

Consider now the following symmetric bilinear form :

$$\forall \rho \in \mathcal{V},\quad \forall \hat{\rho} \in \mathcal{V},\quad a(\rho,\hat{\rho}) = \int_U\int_\Omega L\rho L\hat{\rho} dtdadx + \int_U\int_\omega (\rho - P\rho)\,(\hat{\rho}) - P\hat{\rho}))\, dtdadx. \tag{3.1}$$

According to Lemma 2.2, this symmetric bilinear form is a scalar product on $\mathcal{V}$. Let V be the completion of $\mathcal{V}$ with respect to the related norm:

$$\rho \mapsto \|\rho\|_V = \sqrt{a(\rho,\rho)}. \tag{3.2}$$

The closure of $\mathcal{V}$ is the Hilbert space V.

Remark 3.1. *We have :*

1. *The norm* $\|.\|_V$ *is related to the right side of inequality* (2.14) *while the left member of* (2.14) *leads to the norm*

$$\forall \rho \in \mathcal{V},\quad |\rho|_\theta = \left(\int_Q \frac{1}{\theta^2}|\rho|^2 dtdadx\right)^{\frac{1}{2}}$$

2. *The completion of $\mathcal{V}$ is the weighted Hilbert space usually denoted by $L^2_{\frac{1}{\theta}}$.*

3. *The inequality* (2.14) *shows that*

$$|\rho|_\theta \leq C\|\rho\|_V. \tag{3.3}$$

Let θ be defined by (2.12) *and $h \in L^2(Q)$ be such that $\theta h \in L^2(Q)$. Then, thanks to Cauchy-Schwartz's inequality and* (2.14)*, the following linear form defined on V by :*

$$\rho \longrightarrow \int_U\int_\Omega h\rho ddadx$$

is continuous. Therefore, Lax-Milgram's Theorem [6], allows us to say that, for every function $h \in L^2(Q)$ such that $\theta h \in L^2(Q)$, there exist one and only one solution ρ_θ in V of the variational equation:

$$a(\rho_\theta, \rho) = \int_U\int_\Omega h\rho ddadx \quad \forall \quad \rho \in V. \tag{3.4}$$

In the following we assume that :

$$\begin{array}{rcl} q(T) & \in & L^2([0,A], H^{-2}(\Omega)), \\ q(0) & \in & L^2([0,T], H^{-2}(\Omega)), \\ q(A) & \in & L^2([0,T], H^{-2}(\Omega)). \end{array}$$

Proposition 3.2. *Assume* (1.10) *holds. For $h \in L^2(Q)$ such that $\theta h \in L^2(Q)$, let ρ_θ be the unique solution of* 3.4*,*

$$v_\theta = -\left(\rho_\theta\chi_\omega - P\rho_\theta\right), \tag{3.5}$$

and

$$q_\theta = L\rho_\theta. \tag{3.6}$$

Then, the pair (v_θ, q_θ) is such that (1.11) − (1.13) *holds.*

Proof. We prove that (v_θ, q_θ) is a solution of (1.11) − (1.13). According to (3.5), we have $\rho_\theta \in V$. Consequently $q_\theta \in L^2(Q)$ and since $P\rho_\theta \in Y_\lambda$, the function $v_\theta = -\left(\rho_\theta\chi_\omega - P\rho_\theta\right) \in Y_\lambda^\perp$. Next, replacing $L\rho_\theta$ by $q_\theta \in L^2(Q)$ and $-\left(\rho_\theta\chi_\omega - P\rho_\theta\right)$ by v_θ in 3.4, we obtain

$$\int_U\int_\Omega q_\theta L\rho dtdadx - \int_U\int_\omega v_\theta\left(\rho - P\rho\right)dtdadx = \int_U\int_\Omega h\rho dtdadx, \quad \forall \rho \in V.$$

Since $P\rho \in Y_\lambda$ and $v_\theta \in Y_\lambda$, this latter equality is reduced to

$$\int_U\int_\Omega q_\theta L\rho dtdadx = \int_U\int_\Omega h\rho dtdadx + \int_U\int_\omega v_\theta\rho dtdadx, \quad \forall \rho \in V. \tag{3.7}$$

In the duality frame $\mathcal{D}(Q)$, $\mathcal{D}'(Q)$ (3.7) means that

$$L^* q_\theta = h + v_\theta\chi_\omega \;\text{ in }\; \mathcal{D}'(Q). \tag{3.8}$$

Besides $h + v_\theta\chi_\omega \in L^2(Q)$, then $L^* q_\theta \in L^2(Q)$.

Since $q_\theta \in L^2(Q)$ and $\Delta q_\theta \in H^{-1}(U, L^2(\Omega))$ and by the above Remark $q_\theta\mid_{U\times\Gamma} \in H^{-1}(U, H^{-\frac{1}{2}}(\Gamma))$. Similarly, since $q_\theta \in L^2(Q)$ and $\frac{\partial q_\theta}{\partial t} + \frac{\partial q_\theta}{\partial a} \in L^2(U, H^{-2}(\Omega))$, $q_\theta(0,a,x) \in L^2([0,A], H^{-2}(\Omega))$,

$q_\theta(T,a,x) \in L^2([0,A],H^{-2}(\Omega))$; $q_\theta(t,0,x) \in L^2([0,T],H^{-2}(\Omega))$, $q_\theta(t,0,x) \in L^2([0,T],H^{-2}(\Omega))$. Taking into account (3.8), integrate by parts

$$\begin{aligned}
\forall \rho \in \mathcal{V}, \quad & \int_U\int_\Omega q_\theta L\rho dtdadx + \int_U \left\langle q_\theta, \frac{\partial\rho}{\partial\nu}\right\rangle_{H^{-\frac{1}{2}}(\Gamma_1),H^{\frac{1}{2}}(\Gamma_1)} dtda + \int_U \left\langle \frac{\partial q_\theta}{\partial\nu}, \rho\right\rangle_{H^{-\frac{1}{2}}(\Gamma\setminus\Gamma_1),H^{\frac{1}{2}}(\Gamma\setminus\Gamma_1)} dtda \\
+ \quad & \int_0^T \left[\langle q_\theta(t,0,.),\rho(t,0,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(t,A,.),\rho(t,A,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}\right] dt \\
+ \quad & \int_0^T \left[\langle q_\theta(0,a,.),\rho(0,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(T,a,.),\rho(T,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}\right] da \\
= \quad & \int_\Omega (h + v_\theta\chi_\omega)\rho dtdadx
\end{aligned}$$

By (3.7), since $\mathcal{V} \in V$, it follows

$$\begin{aligned}
\forall \rho \in \mathcal{V}, \quad & \int_U \left\langle q_\theta, \frac{\partial\rho}{\partial\nu}\right\rangle_{H^{-\frac{1}{2}}(\Gamma_1),H^{\frac{1}{2}}(\Gamma_1)} dtda + \int_U \left\langle \frac{\partial q_\theta}{\partial\nu}, \rho\right\rangle_{H^{-\frac{1}{2}}(\Gamma\setminus\Gamma_1),H^{\frac{1}{2}}(\Gamma\setminus\Gamma_1)} dtda \\
+ \quad & \int_0^T \left[\langle q_\theta(t,0,.),\rho(t,0,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(t,A,.),\rho(t,A,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}\right] dt \\
+ \quad & \int_0^T \left[\langle q_\theta(0,a,.),\rho(0,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(T,a,.),\rho(T,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}\right] da \\
= \quad & 0
\end{aligned}$$

Then, successively, we get $q_\theta = 0$ on Σ_1, $\frac{\partial q_\theta}{\partial\nu} = 0$ on $\Sigma \setminus \Sigma_1$; $q_\theta(0,a,x) = 0$ and $q_\theta(T,a,x) = 0$ in Q_A; $q_\theta(t,0,x) = 0$ and $q_\theta(t,A,x) = 0$ in Q_T.
Since $q_\theta(t,0,x) = 0$ we have

$$L^*q_\theta = \beta q(t,0,x) + h_0\chi_{\mathcal{O}} + \mathrm{w}\chi_\omega$$

Hence the proof is completed.

□

Proposition 3.3. *Under the assumptions of the Proposition 3.3, there exists a control variable v such that the pair (v,q) satisfies* (1.11) − (1.14)*. Moreover, we can get a unique control $\hat{v_\theta}$ such that* (1.15) *holds.*

Proof. We have proved in Proposition 3.1 that (v_θ, q_θ) satisfies (1.11)–(1.14). Consequently, the set ε of control variables $v \in L^2(U\times\omega)$ such that $(v, q(t,a,x,v))$ verifies (1.11)–(1.14) is non-empty. Moreover, the adapted observability inequality (2.14) shows that the choice of the scalar product on $\mathcal{V}$ is not unique. Thus, proceeding as in Proposition 3.1, we can construct infinitely many control functions v that belong to ε. It is then clear that ε is a non-empty closed convex subset of $L^2(U\times\omega)$. Therefore, there exists a unique control variable $\hat{v_\theta}$ of minimal norm in $L^2(U\times\omega)$ such that $(\hat{v_\theta}, \hat{q_\theta} = q(t,a,x,\hat{v_\theta}))$ solves (1.11)–(1.14).

□

3.2 Proof of Theorem 1.5

In this subsection, we are concerned with the proof of Theorem 1.5. That is, the optimality system for the control $\hat{v_\theta}$ such that the pair $(\hat{v_\theta}, \hat{q_\theta})$ satisfies (1.11) − (1.14). As a classical way to derive this optimality system is the method of penalization due to J.L.Lions [11], the proof of theorem 1.2

requires some preliminary results.

Let $\epsilon > 0$. We define the functional

$$J_\epsilon(v,q) = \frac{1}{2}\|v\|^2_{L^2(U\times\omega)} + \frac{1}{2\epsilon}\| - \frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q - \beta q(t,0,x) - h - v\chi_\omega\|^2_{L^2(Q)}, \tag{3.9}$$

for any pair (v,q) such that

$$\begin{cases} v \in Y_\lambda^\perp, q \in L^2(Q), \\ -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q - \beta q(t,0,x) \in L^2(Q), \\ q = 0 \ \text{ on } \ \Sigma_1, \ \ \frac{\partial q}{\partial \nu} = 0 \ \text{ on } \ \Sigma \setminus \Sigma_1, \\ q(T,a,x) = 0 \ \text{ in } \ Q_A, \ \ q(t,A,x) = 0 \ \text{ in } \ Q_T, \\ q(0,a,x) = 0 \ \text{ in } \ Q_A. \end{cases} \tag{3.10}$$

and we consider the minimization problem

$$\inf\{J_\epsilon(v,q) \mid (v,q) \ \text{ subject } \ to \ \ (3.10)\} \tag{3.11}$$

Proposition 3.4. *Under the assumptions of proposition 3.3 , the problem* (3.11) *has an optimal solution. In other words, there exists a unique pair* (v_ϵ, q_ϵ) *such that*

$$J_\epsilon(v_\epsilon, q_\epsilon) = \inf\{J_\epsilon(v,q) \mid (v,q) \ \text{ subject } \ to \ \ (3.10)\} \tag{3.12}$$

Proof. Let (v_n, q_n) be a minimizing sequence satisfying (3.10). The sequence $(J_\epsilon(v_n,q_n))_n$ is bounded from above

$$J_\epsilon(v_n, q_n) \leq \gamma(\epsilon), \tag{3.13}$$

then

$$\begin{cases} \|v_n\|_{L^2(U\times\omega)} \leq C(\epsilon), \\ \left\| -\frac{\partial q_n}{\partial t} - \frac{\partial q_n}{\partial a} - \Delta q_n + \mu q_n - \beta q_n(t,0,x) - h - v_n\chi_\omega \right\|_{L^2(Q)} \leq \sqrt{\epsilon} C(\epsilon). \end{cases} \tag{3.14}$$

There is some subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, such that

$$v_n \rightharpoonup v_\epsilon \ \text{ weakly in } \ L^2(U\times\omega). \tag{3.15}$$

As a consequence (3.10) the sequence $(q_n)_n$ is bounded

$$\|q_n\|_{L^2(Q)} \leq C. \tag{3.16}$$

There is some subsequence of $(q_n)_n$, still denoted by $(q_n)_n$ such that

$$q_n \rightharpoonup q_\epsilon \ \text{ weakly in } \ L^2(Q). \tag{3.17}$$

Then

$$\lim \ \inf J_\epsilon(v_n, q_n) \geq (J_\epsilon(v_\epsilon, q_\epsilon). \tag{3.18}$$

We deduce that (v_ϵ, q_ϵ) is a unique optimal control, from the strict convexity of J_ϵ ☐

Proposition 3.5. *The assumptions are as in Proposition 3.3. Then, the pair* (v_ϵ, q_ϵ) *is optimal solution of the problem* (3.12) *if and only if there exists a function* ρ_ϵ *such that* $(v_\epsilon, q_\epsilon, \rho_\epsilon) \in L^2(U \times \omega) \times L^2(Q) \times V$ *satisfies the following approximate optimality system:*

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x) + h + v_\epsilon \chi_\omega + \epsilon \rho_\epsilon \quad \text{in} \quad Q, \\ q = 0 \quad \text{on} \quad \Sigma_1, \\ \frac{\partial q}{\partial \nu} = 0 \quad \text{on} \quad \Sigma \setminus \Sigma_1, \\ q(T,a,x) = 0 \quad \text{in} \quad Q_A, \\ q(t,A,x) = 0 \quad \text{in} \quad Q_T; \end{cases} \tag{3.19}$$

$$q(0,a,x) = 0 \quad \text{in} \quad Q_A. \tag{3.20}$$

$$\begin{cases} \frac{\partial \rho_\epsilon}{\partial t} + \frac{\partial \rho_\epsilon}{\partial a} - \Delta \rho_\epsilon + \mu \rho_\epsilon = 0 \quad \text{in} \quad Q, \\ \rho_\epsilon = 0 \quad \text{on} \quad \Sigma_1, \\ \frac{\partial \rho_\epsilon}{\partial \nu} = 0 \quad \text{on} \quad \Sigma \setminus \Sigma_1, \\ \rho_\epsilon(t,0,x) = \displaystyle\int_0^A \beta(t,a,x)\rho_\epsilon(t,a,x)da \quad \text{in} \quad Q_T; \end{cases} \tag{3.21}$$

$$v_\epsilon = -(\rho_\epsilon \chi_\omega - P\rho_\epsilon) \in Y_\lambda^\perp \tag{3.22}$$

Proof. Express the Euler-Lagrange optimality condition which characterize (v_ϵ, q_ϵ). For any (v, φ) such that (3.10) the following holds

$$\begin{aligned} &\int_U \int_\omega v_\epsilon v dt da dx + \\ &\tfrac{1}{\epsilon} \int_Q \left(-\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q - \beta q(t,0,x) - h - v_\epsilon \chi_\omega \right) \\ &\times \left(-\tfrac{\partial \varphi}{\partial t} - \tfrac{\partial \varphi}{\partial a} - \Delta \varphi + \mu \varphi - \beta \varphi(t,0,x) - v \chi_\omega \right). \end{aligned} \tag{3.23}$$

Define the adjoint state

$$\rho_\epsilon = -\frac{1}{\epsilon} \left(-\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon - \beta q_\epsilon(t,0,x) - h - v_\epsilon \chi_\omega \right). \tag{3.24}$$

Then (3.19) holds.

For any (v, φ) such that (3.10), (3.23) becomes

$$\int_U \int_\omega v_\epsilon v dt da dx + \int_Q \rho_\epsilon \left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - \Delta \varphi + \mu \varphi - \beta \varphi(t,0,x) - v \chi_\omega \right) dt da dx = 0. \tag{3.25}$$

Integrate by parts in (3.25). As a consequence the couple (v_ϵ, q_ϵ) is shown to satisfy

$$\begin{cases} \frac{\partial \rho_\epsilon}{\partial t} + \frac{\partial \rho_\epsilon}{\partial a} - \Delta \rho_\epsilon + \mu \rho_\epsilon = 0 \quad \text{in} \quad Q, \\ \rho_\epsilon = 0 \quad \text{on} \quad \Sigma_1, \\ \frac{\partial \rho_\epsilon}{\partial \nu} = 0 \quad \text{on} \quad \Sigma \setminus \Sigma_1, \\ \rho_\epsilon(t,0,x) = \displaystyle\int_0^A \beta(t,a,x)\rho_\epsilon(t,a,x)da \quad \text{in} \quad Q_T; \end{cases} \tag{3.26}$$

and

$$\int_U \int_\omega (v_\epsilon + \rho_\epsilon)\, dtdadx = 0, \quad \forall \;\; v \in Y_\lambda^\perp. \tag{3.27}$$

Hence $v_\epsilon + \rho_\epsilon \chi_\omega \in Y_\lambda^\perp$. Since $v_\epsilon \in Y_\lambda^\perp$ then $v_\epsilon + \rho_\epsilon \chi_\omega = P(v_\epsilon + \rho_\epsilon \chi_\omega) = P\rho_\epsilon$ and thus

$$v_\epsilon = -(\rho_\epsilon \chi_\omega - P\rho_\epsilon). \tag{3.28}$$

Hence the assertion follows. □

Remark 3.6. *There is no available information concerning $\rho_\epsilon(t, A, x)$ in Q_T, $\rho_\epsilon(0, a, x)$ in Q_A, $\rho_\epsilon(T, a, x)$ in Q_A.*

Proposition 3.7. *Let $(v_\epsilon, q_\epsilon, \rho_\epsilon)$ be defined as in Proposition 3.5. Then there exists a constant $C > 0$ independent on ϵ such that*

$$\|q_\epsilon\|_{L^2(Q)} \le C, \tag{3.29}$$

$$\|\rho\epsilon - P\rho_\epsilon\|_{L^2(U\times\omega)} \le C, \tag{3.30}$$

$$\|\rho\epsilon\|_{L^2(U\times\omega)} \le C, \tag{3.31}$$

$$\|\rho\epsilon\|_V \le C. \tag{3.32}$$

Proof. From (3.14), we have

$$\left\| -\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon - \beta q_\epsilon(t, 0, x) - h - v_\epsilon \chi_\omega \right\|_{L^2(Q)} \le C\sqrt{\epsilon}, \tag{3.33}$$

$$\|v_\epsilon\|_{L^2(U\times\omega)} \le C. \tag{3.34}$$

Since q_ϵ verifies (3.10), we derive from (3.33), the relation (3.29). From (3.22) and (3.34), we obtain (3.30). Then as $L\rho_\epsilon = 0$, using the definition of the norm on V given by (3.2), we have (3.32) in one hand.

On the over hand, since $\rho_\epsilon \in \mathcal{V}$, applying the observability inequality (2.17) to ρ_ϵ, we have $\left\|\frac{1}{\theta}\rho_\epsilon\right\|_{L^2(U\times\omega)} \le C$. Therefore, using (3.30) and the fact that $\frac{1}{\theta}$ is in $L^{\inf}(Q)$, we deduce that $\left\|\frac{1}{\theta}P\rho_\epsilon\right\|_{L^2(U\times\omega)} \le C$. Since $P\rho_\epsilon$ is in Y_λ which is finite dimensional, we have $\|P\rho_\epsilon\|_{L^2(U\times\omega)} \le C$. Hence using again (3.30), we obtain estimate (3.31). □

Proof. of Theorem 1.5 .

We proceed in three steps :

Step 1. We study the convergence of $(v_\epsilon, q_\epsilon)_\epsilon$.
According to (3.34) and (3.29) we can extract sub-sequences, still denoted $(q_\epsilon)_\epsilon$ and $(v_\epsilon)_\epsilon$ such that

$$v_\epsilon \rightharpoonup v_0 \;\text{ Weakly in }\; L^2(U\times\omega), \tag{3.35}$$

$$q_\epsilon \rightharpoonup q_0 \;\text{ Weakly in }\; L^2(Q). \tag{3.36}$$

And, as (v_ϵ) belong to $Y_\lambda^\perp$ which is a closed vector subspace of $L^2(U\times\omega)$, we have

$$v_0 \in Y_\lambda^\perp. \tag{3.37}$$

From (3.36), we have $q_\epsilon \rightharpoonup q_0$ Weakly in $\mathcal{D}'(Q)$. and by the weak continuity of the operator L^* in $\mathcal{D}'(Q)$ it follows $L^* q_\epsilon \rightharpoonup L^* q_0$ Weakly in $\mathcal{D}'(Q)$. Moreover the traces functions are continuous, then the pair (v_0, q_0) satisfies the system

$$\begin{cases} -\frac{\partial q_0}{\partial t} - \frac{\partial q_0}{\partial a} - \Delta q_0 + \mu q_0 = \beta q_0(t,0,x) + h + v_0\chi_\omega \text{ in } Q, \\ q_0 = 0 \quad \text{on } \Sigma_1, \\ \frac{\partial q_0}{\partial \nu} = 0 \quad \text{on } \Sigma \setminus \Sigma_1, \\ q_0(T,a,x) = 0 \text{ in } Q_A, \\ q_0(t,A,x) = 0 \text{ in } Q_T, \\ q_0(0,a,x) = 0 \text{ in } Q_A. \end{cases} \tag{3.38}$$

$$q(0,a,x) = 0 \text{ in } Q_A. \tag{3.39}$$

Step 2. We prove that $(v_0, q_0 = q(t,a,x,v_0)) = (\hat{v}_\theta, \hat{q}_\theta = q(t,a,x,\hat{v}\theta))$. From the expression of J_ϵ given by (3.9), we can write

$$\frac{1}{2}\|v_\epsilon\|^2_{L^2(U\times\omega)} \le J_\epsilon(v_\epsilon, q_\epsilon).$$

Since $(\hat{v}_\theta, \hat{q}_\theta)$ satisfies (1.11) − (1.13)(or equivalently verifies (3.10)), this latter inequality becomes

$$\frac{1}{2}\|v_\epsilon\|^2_{L^2(U\times\omega)} \le J_\epsilon(v_\epsilon, q_\epsilon) \le \frac{1}{2}\|\hat{v}_\theta\|^2_{L^2(U\times\omega)}. \tag{3.40}$$

Then using (3.35) while passing to the limit in (3.40), we obtain

$$\frac{1}{2}\|v_0\|^2_{L^2(U\times\omega)} \le \lim_{\epsilon\to 0}\inf J_\epsilon(v_\epsilon, q_\epsilon) \le \frac{1}{2}\|\hat{v}_\theta\|^2_{L^2(U\times\omega)}.$$

Consequently,

$$\|v_0\|_{L^2(U\times\omega)} \le \|\hat{v}_\theta\|_{L^2(U\times\omega)},$$

and thus,

$$\|v_0\|_{L^2(U\times\omega)} = \|\hat{v}_\theta\|_{L^2(U\times\omega)}.$$

Hence, $v_0 = \hat{v}_\theta$ and since (3.38) has a unique solution, it follows that $q_0 = \hat{q}_\theta$.

Step 3. According to the inequalities (3.31) and (3.32), we can extract a subsequence, still denoted $(\rho_\epsilon)_\epsilon$ such that

$$\rho_\epsilon \rightharpoonup \hat{\rho}_\theta \text{ Weakly in } L^2(U\times\omega), \tag{3.41}$$

$$\rho_\epsilon \rightharpoonup \hat{\rho}_\theta \text{ Weakly in } V. \tag{3.42}$$

As P is a compact operator, we deduce from (3.41) that

$$P\rho_\epsilon \rightharpoonup P\hat{\rho}_\theta \text{ strongly in } L^2(U\times\omega). \tag{3.43}$$

Therefore, combining (3.41) and (3.43), we get

$$v_\epsilon = \rho_\epsilon\chi_\omega - P\rho \rightharpoonup \hat{v}_\theta = \hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta \text{ Weakly in } L^2(U\times\omega).$$

Thus, we have proved that there exists θ given by (2.12) such that for a given $h \in L^2(Q)$ with $\theta h \in L^2(Q)$, the unique pair $(\hat{v}_\theta, \hat{q}_\theta)$ satisfies (1.11) – (1.14) with $\hat{v}_\theta = \hat{\rho}_\theta \chi_\omega - P\hat{\rho}_\theta$, and where $\hat{\rho}_\theta$ is a solution of (1.17). Since the function h defined by (2.9) belongs to $L^2(Q)$ if $\theta h \in L^2(Q)$, the proof of Theorem 1.2 is complete.

□

4 Expression of the sentinel with given sensitivity and identification of parameter λ_i

We can now give the expression of sentinel S defined by (1.3) – (1.6) and identify the parameters λ_i

4.1 Expression of the sentinel with given sensitivity

We consider the results obtain in the previous section and we assume that h given by (2.9) and θ given by (2.12) are such that $\theta h \in L^2(U \times \mathcal{O})$. Let $(\hat{\rho}_\theta, \hat{v}_\theta)$ be defined as in Theorem 1.5. Since $\hat{v}_\theta = -(\hat{\rho}_\theta \chi_\omega - P\hat{\rho}_\theta)$ realizes the minimum in $L^2(U \times \omega)$ among all controls v such that the pair (v, q) satisfies (1.11) – (1.14), using (2.8), we deduce that $\mathrm{w} = \mathrm{w}_0 + \hat{v}_\theta = \mathrm{w}_0 - (\hat{\rho}_\theta \chi_\omega - P\hat{\rho}_\theta)$. Consequently, replacing w by its expression in (1.3), the function S becomes:

$$S(\lambda, \tau) = \int_U \int_{\mathcal{O}} h_0 y(\lambda, \tau) dt da dx + \int_U \int_\omega (\mathrm{w}_0 - (\hat{\rho}_\theta \chi_\omega - P\hat{\rho}_\theta)) y(\lambda, \tau) dt da dx, \tag{4.1}$$

and (w, S) is such that (1.4) – (1.6) hold.

4.2 Identification of parameter λ_i

y_0 is the solution of the problem (1.1) when $\lambda = 0$ and $\tau = 0$. Hence, from (4.1) we have

$$S(0,0) = \int_U \int_{\mathcal{O}} h_0 y_0 dt da dx + \int_U \int_\omega (\mathrm{w}_0 - (\hat{\rho}_\theta \chi_\omega - P\hat{\rho}_\theta)) y_0 dt da dx = 0.$$

Next, using (1.4), we obtain

$$S(\lambda, \tau) - S(0,0) \simeq \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0) \quad \text{for} \quad \lambda_i \quad \text{and} \quad \tau \quad \text{small}.$$

Since get at our disposal the observation y_{obs}, we get

$$S(\lambda, \tau) - S(0,0) = \int_U \int_{\mathcal{O}} h_0 (y_{obs} - y_0) dt da dx + \int_U \int_\omega \mathrm{w}(y_{obs} - y_0) dt da dx = 0.$$

Thus, we also have the following information:

$$\sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0) \simeq \int_U \int_{\mathcal{O}} h_0 (y_{obs} - y_0) dt da dx + \int_U \int_\omega \mathrm{w}(y_{obs} - y_0) dt da dx,$$

which, using (1.5)

$$\sum_{i=1}^{M} \lambda_i c_i \simeq \int_U \int_{\mathcal{O}} h_0(y_{obs} - y_0) dt da dx + \int_U \int_{\omega} \mathrm{w}(y_{obs} - y_0) dt da dx.$$

Now, fixing $i \in \{1, ..., M\}$ and choosing $c_i \neq 0$ and $c_j = 0$, for all $j \in \{1, ..., M\}$ with $j \neq i$, we get this estimate of the parameter λ_i

$$\lambda_i \simeq \frac{1}{c_i} \int_U \int_{\mathcal{O}} h_0(y_{obs} - y_0) dt da dx + \int_U \int_{\omega} \mathrm{w}(y_{obs} - y_0) dt da dx.$$

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