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Multi-objective optimal control of the combined dynamics of a fanatical insurgency and narcoterrorism in the Sahel

Mathieu Romaric POODA [1], **Yacouba SIMPORE** [2] **and Oumar TRAORE** [3]

[1] Laboratoire de Sciences et Technologies, Université Thomas SANKARA,
12 BP 417 Ouagadougou 12, e-mail : math7roma8@gmail.com
[2] LAboratoire de Mathématiques et d'Informatique, Université Joseph KI-ZERBO,
03 BP 7021 Ouagadougou 03, e-mail : simplesaint@gmail.com
[3] Laboratoire de Sciences et Technologies, Université Thomas SANKARA,
12 BP 417 Ouagadougou 12, e-mail : oumar.traore@uts.bf

Abstract : In this study, we formulate and investigate a multi-objective control problem aimed at eradicating fanatical insurgent armed groups, narcoterrorists and banditry in the Sahel. The aim is to identify different control scenarios and integrate them into a model of the combined dynamics of a fanatical insurgency and narcoterrorism. We analyze the effectiveness of these control strategies using an optimality study based on Pontryagin's maximum principle. Then, we perform numerical simulations to assess the impact of these control measures on the evolution of the combined dynamics. This research has the potential to contribute to the fight against violent extremism and to promote stability in the Sahel states.

Keywords : fanatical insurgency, narcoterrorism, optimal control, numerical simulation.
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1 Introduction

The expansion of fanatical insurgent groups since the Libyan crisis of 2011, and the rise of drug traffickers in the Sahel, have transformed this region into a zone of instability and the epicenter of violence in Africa [2]. This deadly combination of fanaticism, terrorism and drug trafficking

1. Corresponding author : Mathieu Romaric POODA

represents a complex and constantly evolving challenge that requires an integrated and strategic approach. In this quest for security and stability in the region, mathematical modeling is emerging as a powerful tool for analyzing, anticipating, and optimizing policies to control and counter violent extremism and drug trafficking.

Optimal control in this context refers to a variety of options and strategies based on mathematical models. These options include resource allocation for security operations, military operational planning, regional coordination of actions, information management, border management, early detection of threats, rehabilitation of radicalized individuals and prevention of recruitment, among others. Mathematical modeling provides a framework for evaluating the interaction of these options and optimizing them. It is widely recognized that mathematical modeling plays an essential role in understanding and solving complex problems, including studies of migration and crowd behavior [5], [22], [16], crime [10], [11], [19], [14], [12], [13], [21], [17], gang membership [26], [1], [6], the dynamics of war [3], [9], [8], and research into the transmission dynamics of fanatical behavior [7].

This work is divided into several distinct sections, each exploring in detail different aspects of our research. In section 2, we formulate a model of the combined dynamics of a fanatical insurgency and narcoterrorism in the Sahel. In Section 3, we carry out a theoretical analysis of the existence, uniqueness and positivity of solutions to the model's equation. In section 4, we formulate our control strategy, examining the principles and methodologies needed to create practical approaches to mitigating the risks associated with the convergence of fanatical armed insurgency and narcoterrorism. In Section 5, we establish the existence and characterization of control optimality, providing a sound theoretical basis for our approach. In Section 6, we perform numerical simulations to implement our control strategies and evaluate their performance in realistic scenarios. Finally, in Section 7, we present our conclusions, summarizing our main results and discussing the implications of this research.

2 Model formulation

In this section we formulate the model of the combined dynamics of a fanatical insurgency and narco terrorism in the Sahel. This part is essentially devoted to the description of the variables which intervene in the model as well as the different parameters. To enhance clarity, the model divides the total population (N) into two groups. The first group (D) consists of sub-populations at the core of fanatical ideology, including vulnerable individuals (S), semi-fanatical followers (E), fully committed fanatics (F), and terrorists involved in extremist activities (T). It is important to note that the hierarchy within the fanatical subpopulation is defined by different levels of commitment, with F representing the highest level. The second group (G) is composed of various subgroups, including non-combatant civilians with an extremist ideology (C), homeland defense volunteers (V), defense and security forces (A), personnel disbarred from these forces (R), brigands (B), and narcoterrorist cartels (K). The class $I = A + V + B + T + K$ is made up of armed individuals engaged in combat. The dynamics of these classes and their interactions are illustrated in Figure 2. All parameters of system (2.1) are assumed to be non-negative, and they are enumerated and defined in Table 1.

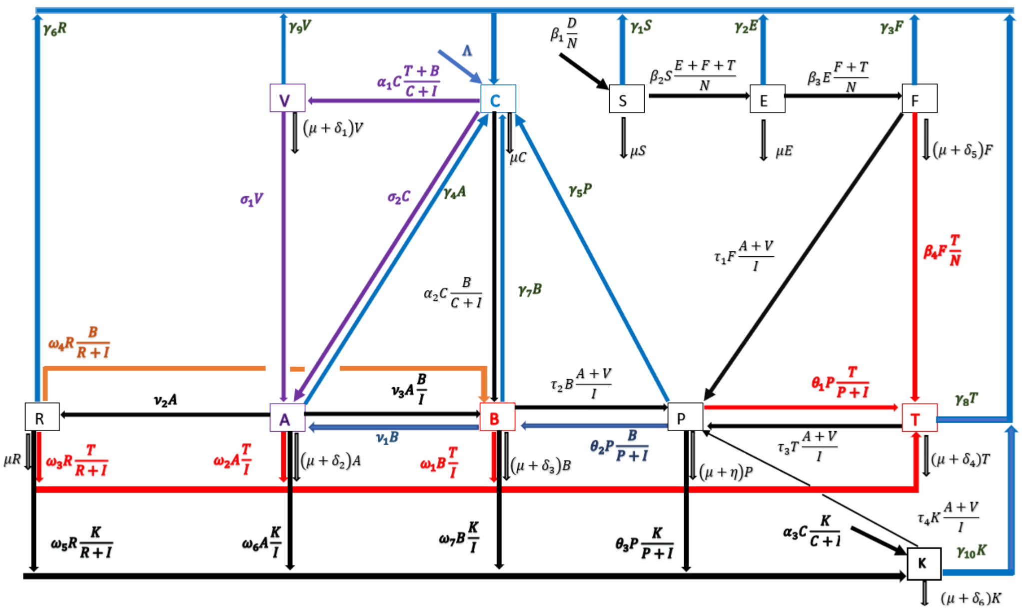

FIGURE 1 – Diagram of the combined dynamics of fanatical insurgency and narcoterrorism in the Sahel

The model equation is given by the following system :

$$
\left\{
\begin{array}{lcl}
\frac{dC}{dt} & = & \Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V + \gamma_{10} K - \left(\pi_1 \frac{D}{N} + \alpha_1 \frac{T+B}{C+I} + \alpha_2 \frac{B}{C+I} + \alpha_3 \frac{K}{C+I} + \sigma_2 + \mu\right) C \\
\frac{dR}{dt} & = & \nu_2 A - \left(\pi_5 \frac{D}{N} + \omega_3 \frac{T}{R+I} + \omega_4 \frac{B}{R+I} + \omega_5 \frac{K}{R+I} + \gamma_6 + \mu\right) R \\
\frac{dA}{dt} & = & \sigma_1 V + \sigma_2 C + \nu_1 B - \left(\pi_3 \frac{D}{N} + \nu_3 \frac{B}{I} + \omega_2 \frac{T}{I} + \omega_6 \frac{K}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B+K}{I}\right) A \\
\frac{dV}{dt} & = & \alpha_1 C \frac{T+B}{C+I} - \left(\pi_2 \frac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B+K}{I}\right) V \\
\frac{dS}{dt} & = & \left(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P + \pi_7 K\right) \frac{D}{N} - (\beta_2 \frac{E+F+T}{N} + \gamma_1 + \mu) S \\
\frac{dE}{dt} & = & \beta_2 S \frac{E+F+T}{N} - (\beta_3 \frac{F+T}{N} + \gamma_2 + \mu) E \\
\frac{dF}{dt} & = & \beta_3 E \frac{F+T}{N} - \left(\gamma_3 + \tau_1 \frac{A+V}{F+I} + \beta_4 \frac{T}{N} + \mu + \zeta_3 \frac{A+V}{F+I}\right) F \\
\frac{dB}{dt} & = & \alpha_2 \frac{BC}{C+I} + \omega_4 \frac{BR}{R+I} + \nu_3 \frac{BA}{I} + \theta_2 \frac{BP}{P+I} - \left(\pi_4 \frac{D}{N} + \omega_1 \frac{T}{I} + \omega_7 \frac{K}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4 \frac{A+V}{I}\right) B \\
\frac{dT}{dt} & = & \beta_4 F \frac{T}{N} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_5 \frac{A+V}{I}\right) T \\
\frac{dP}{dt} & = & \tau_1 F \frac{A+V}{F+I} + \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} + \tau_4 K \frac{A+V}{I} - \left(\pi_6 \frac{D}{N} + \theta_1 \frac{T}{P+I} + \theta_2 \frac{B}{P+I} + \theta_3 \frac{K}{P+I} + \gamma_5 + \mu + \eta\right) P \\
\frac{dK}{dt} & = & \alpha_3 C \frac{K}{C+I} + \omega_5 R \frac{K}{R+I} + \omega_6 A \frac{K}{I} + \omega_7 B \frac{K}{I} + \theta_3 P \frac{K}{P+I} - \left(\pi_7 \frac{D}{N} + \tau_4 \frac{A+V}{I} + \gamma_{10} + \mu + \zeta_6 \frac{A+V}{I}\right) K
\end{array}
\right. \tag{2.1}
$$

with the non-negative initial conditions given by :

$$C(0) > 0; S(0) \geq 0; E(0) \geq 0; F(0) \geq 0; V(0) \geq 0; A(0) > 0; R(0) \geq 0; B(0) \geq 0; P(0) \geq 0; T(0) \geq 0; K(0) \geq 0; N(0) \leqslant \frac{\Lambda}{\mu}. \tag{2.2}$$

TABLE 1 – Parameters of the combined dynamic model (2.1) of fanatical insurgency and narcoterrorism in the Sahel

Parameter	Description
Λ	the population renewal rate
γ_1	the recovery rate or return to normal civilian life for individuals in the class S
γ_2	the recovery rate or return to normal civilian life for individuals in the class E
γ_3	the recovery rate or return to normal civilian life for individuals in the class F
γ_4	the recovery rate or return to normal civilian life for individuals in the class A
γ_5	the recovery rate or return to normal civilian life for individuals in the class P
γ_6	the recovery rate or return to normal civilian life for individuals in the class R
γ_7	the recovery rate or return to normal civilian life for individuals in the class B
γ_8	the recovery rate or return to normal civilian life for individuals in the class T
γ_9	the recovery rate or return to normal civilian life for individuals in the class V
γ_{10}	the recovery rate or return to normal civilian life for individuals in the class K
π_1	the ability of the fanatical D core to recruit and indoctrinate or attract the class C
π_2	the ability of the fanatical D core to recruit and indoctrinate or attract the class V
π_3	the ability of the fanatical D core to recruit and indoctrinate or attract the class A
π_4	the ability of the fanatical D core to recruit and indoctrinate or attract the class R
π_5	the ability of the fanatical D core to recruit and indoctrinate or attract the class B
π_6	the ability of the fanatical D core to recruit and indoctrinate or attract the class P
π_7	the ability of the fanatical D core to recruit and indoctrinate or attract the class K
θ_1	the ability to recruit an individual from class P into class T
θ_2	the ability to recruit an individual from class P into class B
θ_3	the ability to recruit an individual from class P into class K
η	the probability of dying in prison as a result of torture or detention conditions
ζ_1	The fighting strength or firepower of individuals of classes B, T and K over individuals of class V
ζ_2	The fighting strength or firepower of individuals of classes B, T and K on individuals of class A
ζ_3	The fighting strength or firepower of individuals of classes A and V on individuals of class B
ζ_4	The fighting strength or firepower of individuals of classes A and V on individuals of class T
ζ_5	The fighting strength or firepower of individuals of classes A and V on individuals of class F
ζ_6	The fighting strength or firepower of individuals of classes A and V on individuals of class K
μ	natural mortality rate
ν_1	the probability of recruitment into class A of individuals from class B following a malfunction
ν_2	the write-off or dismissal rate in class A
ν_3	the ability to recruit A individuals into the B class
τ_1	the ability of individuals from classes A and V to arrest an individual from class F
τ_2	the ability of individuals from classes A and V to arrest an individual from class B
τ_3	the ability of individuals from classes A and V to arrest an individual from class T
τ_4	the ability of individuals from classes A and V to arrest an individual from class K
β_2	the strength of conversion from class S to class E
β_3	the strength of conversion from class E to class F
β_4	the strength of conversion from class F to class T
σ_1	the rate of recruitment into the A class of individuals from the V class
σ_2	the rate of recruitment into the A class of individuals from the C class
α_1	the strength of determination to defend one's homeland
α_2	the power of attraction or recruitment into the B class of individuals from class C
α_3	the power of attraction or recruitment into the K class of individuals from the C class
ω_1	the power of attraction or recruitment into the T class of individuals from the B class
ω_2	the power of attraction or recruitment into the T class of individuals from the A class
ω_3	the power of attraction or recruitment into the T class of individuals from the R class
ω_4	the power of attraction or recruitment into the B class of individuals from the R class
ω_5	the power of attraction or recruitment into the K class of individuals from the R class
ω_6	the power of attraction or recruitment into the K class of individuals from the A class
ω_7	the power of attraction or recruitment into the K class of individuals from the B class

3 Theoretical analysis of the model

To ensure the realism of system (2.1) in this study, it is essential to establish its well-posedness and appropriate dimensionality. This ensures that all state variables remain positive over time. The subsections of this section focus on proving the existence and uniqueness of solutions, as well as the positivity of the state variables.

3.1 Existence and uniqueness of solution

Given that the system (2.1) is described by a system of non-linear differential equations of first order, we can rewrite it as follows

$$X'(t) = f(X(t)) \tag{3.1}$$

with $X(t)$ a column vector representing the state variables of system (2.1), and $f : \mathbb{R}^{11} \to \mathbb{R}^{11}$ denoting a locally Lipschitz function with respect to X. We establish the existence and uniqueness of the maximum solution of the Cauchy problem associated with the differential equation (2.1) and the initial condition (2.2).

3.2 Positivity of the solutions

Proposition 3.1. ***(Positivity)*** *The positive orthant* $\mathbb{R}^{11}_{\geq 0}$ *remains positively invariant for system* (2.1)*, and the initial condition* (2.2) *guarantees the positivity of solutions for system* (2.1) *at any time* $t > 0$.

Proof : The proof is based on the application of the barrier theorem [4]. For further details, see [25, 24, 27].

4 Formulation of a strategy to control fanatical insurrection and narcoterrorism

Optimal control theory is applied to the model described by the equations (2.1) to deal with fanatical insurgency and brigandage. The introduction of six time-dependent control variables, namely $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$, $u_5(t)$, and $u_6(t)$, each representing a specific strategy against radicalization, violent extremism, and insecurity. Note that for i ranging from 1 to 6, the closer the $u_i(t)$ control value is to 1, the more efficient it is.

(i) The control $u_1(t)$ is a preventive strategy against extremist indoctrination aimed at enhancing social and economic resilience. It focuses on social cohesion, inclusion, and reducing socio-economic inequalities through access to education and employment, as well as resource redistribution policies. By encouraging community bonds, entrepreneurial initiatives, and collaboration with the private sector, this approach seeks to reduce vulnerabilities and create an environment less conducive to extremism.

(ii) The control $u_2(t)$ is a prevention strategy that complements existing efforts by emphasizing a stronger state presence among vulnerable populations. It specifically targets neglected areas and

aims to restore hope to young people. The aim is to give hope to those most vulnerable to radicalization. The state's offer must therefore be more attractive than that of insurgent fanatical groups.

(iii) The control $u_3(t)$ is a deradicalization strategy aimed at creating resilient and inclusive communities that promote peace and peaceful coexistence. This multifaceted approach engages religious and traditional leaders, raises awareness among young people in educational institutions, facilitates reconciliation among citizens, and promotes interfaith and community dialogue. It strives to prevent extremism and support the reintegration of disengaged individuals. By instilling values of tolerance and citizenship from an early age, this strategy builds a strong foundation against radical influences. The objective is to develop resilient and inclusive communities, fostering peace and peaceful coexistence.

(iv) The control $u_4(t)$ refers to the strategy for fighting organized crime, banditry and corruption. It encompasses police operations, the strengthening of territorial networks and the training of defense and security forces. By coordinating the efforts of law enforcement agencies, improving infrastructures and enhancing the capabilities of security personnel, as well as ensuring better territorial networking, this strategy aims to dismantle criminal networks, curb illicit activities and enforce law and order within communities.

(v) The control $u_5(t)$ is a counterterrorism strategy. This strategy places particular emphasis on intelligence development to enhance anticipation capabilities against terrorist attacks. It involves detecting and neutralizing terrorists and their means. It includes monitoring suspicious activities, information exchange between intelligence agencies, and community awareness of radicalization signs. By targeting criminal networks and blocking terrorist financing, the strategy aims to dismantle the support infrastructures of terrorism. Additionally, by addressing root causes such as socio-economic inequalities and strengthening governance, it seeks to create a resilient environment where terrorism cannot thrive. The ultimate goal is to strengthen security and protect communities from the threat of terrorism.

(vi) The control $u_6(t)$ is a series of integrated measures designed to combat drug trafficking and the financing of terrorism, while at the same time tackling drug consumption. It includes strengthening regional cooperation between countries in the region, improving intelligence and security capabilities, and implementing integrated approaches combining military and civilian efforts. It should be noted that measures to combat banditry and terrorism are also effective against narcoterrorism. The particularity of this strategy is that it specifically targets drug traffickers, who are organized by units with greater expertise in this field.

5 Existence and characterization of control optimality

Let

$$c_i(t) = 1 - u_i(t), \qquad \forall i \in \{1, 2, 3, 4, 5, 6\}. \tag{5.1}$$

Consequently, the optimal control model with the above six time variables is given by the following differential equations

$$
\left\{
\begin{array}{lcl}
\frac{dC}{dt} & = & \Lambda^* + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V + \gamma_{10} K - \Big(c_1 \pi_1 \frac{D}{N} + \alpha_1 \frac{T+B}{C+I} + c_4 \alpha_2 \frac{B}{C+I} + c_6 \alpha_3 \frac{K}{C+I} \Big) C \\
\frac{dR}{dt} & = & \nu_2 A - \Big(c_1 \pi_5 \frac{D}{N} + c_5 \omega_3 \frac{T}{R+I} + c_4 \omega_4 \frac{B}{R+I} + c_6 \omega_5 \frac{K}{R+I} + \gamma_6 + \mu \Big) R \\
\frac{dA}{dt} & = & \sigma_1 V + \sigma_2 C + \nu_1 B - \Big(c_1 \pi_3 \frac{D}{N} + c_4 \nu_3 \frac{B}{I} + c_5 \omega_2 \frac{T}{I} + c_6 \omega_6 \frac{K}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B+K}{I} \Big) A \\
\frac{dV}{dt} & = & \alpha_1 C \frac{T+B}{C+I} - \Big(c_1 \pi_2 \frac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B+K}{I} \Big) V \\
\frac{dS}{dt} & = & c_1 \Big(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P + \pi_7 K \Big) \frac{D}{N} - (c_2 \beta_2 \frac{E+F+T}{N} + \gamma_1 + \mu) S \\
\frac{dE}{dt} & = & c_2 \beta_2 S \frac{E+F+T}{N} - (c_3 \beta_3 \frac{F+T}{N} + \gamma_2 + \mu) E \\
\frac{dF}{dt} & = & c_3 \beta_3 E \frac{F+T}{N} - \Big(\gamma_3 + \tau_1 \frac{A+V}{F+I} + c_5 \beta_4 \frac{T}{N} + \mu + \zeta_3 \frac{A+V}{F+I} \Big) F \\
\frac{dB}{dt} & = & c_4 \Big(\alpha_2 C \frac{B}{C+I} + \omega_4 R \frac{B}{R+I} + \nu_3 A \frac{B}{I} + \theta_2 P \frac{B}{P+I} \Big) - \Big(c_1 \pi_4 \frac{D}{N} + c_5 \omega_1 \frac{T}{I} + c_6 \omega_7 \frac{K}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4 \frac{A+V}{I} \Big) B \\
\frac{dT}{dt} & = & c_5 \Big(\beta_4 F \frac{T}{N} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} \Big) - \Big(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_5 \frac{A+V}{I} \Big) T \\
\frac{dK}{dt} & = & c_6 \Big(\alpha_3 C \frac{K}{C+I} + \omega_5 R \frac{K}{R+I} + \omega_6 A \frac{K}{I} + \omega_7 B \frac{K}{I} + \theta_3 P \frac{K}{P+I} \Big) - \Big(c_1 \pi_7 \frac{D}{N} + \tau_4 \frac{A+V}{I} + \gamma_{10} + \mu + \zeta_6 \frac{A+V}{I} \Big) K \\
\frac{dP}{dt} & = & \tau_1 F \frac{A+V}{F+I} + \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} + \tau_4 K \frac{A+V}{I} - \Big(c_1 \pi_6 \frac{D}{N} + c_5 \theta_1 \frac{T}{P+I} + c_4 \theta_2 \frac{B}{P+I} + c_6 \theta_3 \frac{K}{P+I} + \gamma_5 + \mu + \eta \Big) P.
\end{array}
\right. \tag{5.2}
$$

With non-negative initial conditions given by (2.2) and $\Lambda^* = \Lambda - (\sigma_2 + \mu)C$. By applying the barrier theorem [4], we show that all state variables of control system (5.2) remain positive for all times t > 0 and this system can be written in matrix form as follows :

$$X'(t) = g(t, X, c) \tag{5.3}$$

where X is a column vector of state variables, $c = (c_1(t), c_2(t), c_3(t), c_4(t), c_5(t), c_6(t))$ satisfies (5.1), and $g : \mathbb{R} \times \mathbb{R}^{11} \times \mathbb{R}^6 \to \mathbb{R}^{11}$ is a nonlinear function such that (5.2) can be satisfied. The introduction of the six control variables aims to find the optimal solution to minimize the number of individuals in the radical subpopulation or core of violent extremism and fanatical behavior, as well as brigands. Therefore, the objective function for this control problem is given by :

$$\mathcal{J}(u_1, u_2, u_3, u_4, u_5, u_6) = \min_{0 \leqslant u_1, u_2, u_3, u_4, u_5, u_6 \leqslant 1} \int_0^{T_f} \Big(j(t) + \frac{1}{2} k(t) \Big) dt \tag{5.4}$$

where

$$
\begin{aligned}
j(t) &= w_1 S(t) + w_2 E(t) + w_3 F(t) + w_4 B(t) + w_5 T(t) + w_6 P(t) + w_{12} K(t) \\
k(t) &= \Big[w_7 u_1^2(t) + w_8 u_2^2(t) + w_9 u_3^2(t) + w_{10} u_4^2(t) + w_{11} u_5^2(t) + w_{13} u_6^2(t) \Big]
\end{aligned}
$$

with the constants $w_i, i = 1, 2, ..., 13$ are positive weights needed to balance the corresponding terms of the objective function. We choose quadratic costs on the orders, where $\frac{1}{2} w_7 u_1^2(t)$, $\frac{1}{2} w_8 u_2^2(t)$, $\frac{1}{2} w_9 u_3^2(t)$, $\frac{1}{2} w_{10} u_4^2(t)$, $\frac{1}{2} w_{11} u_5^2(t)$, $\frac{1}{2} w_{13} u_6^2(t)$ are the total cost of implementing the preventive measure and the police-military response to manage active cases of armed insurgency and narcoterrorism over the time interval $[0, T_f]$. More precisely, we are looking for the optimum sixfold control

$u^* = \left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*\right)$ is sought such that

$$\mathcal{J}\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*\right) = \min\left\{\mathcal{J}\left(u_1, u_2, u_3, u_4, u_5, u_6\right) : u_1, u_2, u_3, u_4, u_5, u_6 \in \mathcal{U}\right\}, \tag{5.5}$$

where, $\mathcal{U}$ is the non-empty control set defined by

$$\mathcal{U} = \left\{ \left(u_1, u_2, u_3, u_4, u_5, u_6\right) \middle| \begin{array}{c} u_i(t) \text{ is a piecewise continuous function on } [0, T_f] \\ \text{and} \quad 0 \leqslant u_i \leqslant 1, \quad \forall \in t \in [0, T_f], \qquad i = 1, 2, 3, 4, 5, 6 \end{array} \right\} \tag{5.6}$$

Thus, to determine the necessary conditions that the optimal control sixfold must satisfy, we use Pontryagin's maximum principle [23], which transforms the control problem (5.5) subject to model (5.2) into a pointwise minimization problem of a Hamiltonian $\mathcal{H}$. This Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} \quad = \quad & w_1 S + w_2 E + w_3 F + w_4 B + w_5 T + w_6 P + +w_{12} K + \frac{1}{2}\left[w_7 u_1^2(t) + w_8 u_2^2(t) + w_9 u_3^2(t) + w_{10} u_4^2(t) + w_{11} u_5^2(t) + w_{13} u_6^2(t)\right] \\
+ \quad & \lambda_1 \Big[\Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V + \gamma_{10} K - \Big(c_1 \pi_1 \frac{D}{N} + \alpha_1 \frac{T+B}{C+I} + c_4 \alpha_2 \frac{B}{C+I} \\
& + c_6 \alpha_3 \frac{K}{C+I} + \sigma_2 + \mu\Big) C\Big] \\
+ \quad & \lambda_2 \left[\nu_2 A - \left(c_1 \pi_5 \frac{D}{N} + c_5 \omega_3 \frac{T}{R+I} + c_4 \omega_4 \frac{B}{R+I} + c_6 \omega_5 \frac{K}{R+I} + \gamma_6 + \mu\right) R\right] \\
+ \quad & \lambda_3 \left[\sigma_1 V + \sigma_2 C + \nu_1 B - \left(c_1 \pi_3 \frac{D}{N} + c_4 \nu_3 \frac{B}{I} + c_5 \omega_2 \frac{T}{I} + c_6 \omega_6 \frac{K}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B+K}{I}\right) A\right] \\
+ \quad & \lambda_4 \left[\alpha_1 C \frac{T+B}{C+I} - \left(c_1 \pi_2 \frac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B+K}{I}\right) V\right] \\
+ \quad & \lambda_5 \left[c_1 \left(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P + \pi_7 K\right) \frac{D}{N} - (c_2 \beta_2 \frac{E+F+T}{N} + \gamma_1 + \mu) S\right] \\
+ \quad & \lambda_6 \left[c_2 \beta_2 S \frac{E+F+T}{N} - (c_3 \beta_3 \frac{F+T}{N} + \gamma_2 + \mu) E\right] \\
+ \quad & \lambda_7 \left[c_3 \beta_3 E \frac{F+T}{N} - \left(\gamma_3 + \tau_1 \frac{A+V}{F+I} + c_5 \beta_4 \frac{T}{N} + \mu + \zeta_3 \frac{A+V}{F+I}\right) F\right] \\
+ \quad & \lambda_8 \left[c_4 \left(\alpha_2 C \frac{B}{C+I} + \omega_4 R \frac{B}{R+I} + \nu_3 A \frac{B}{I} + \theta_2 P \frac{B}{P+I}\right) - \left(c_1 \pi_4 \frac{D}{N} + c_5 \omega_1 \frac{T}{I} + c_6 \omega_7 \frac{K}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4 \frac{A+V}{I}\right) B\right] \\
+ \quad & \lambda_9 \left[c_5 \left(\beta_4 F \frac{T}{N} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I}\right) - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_5 \frac{A+V}{I}\right) T\right] \\
+ \quad & \lambda_{10} \left[\tau_1 F \frac{A+V}{F+I} + \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} + \tau_4 K \frac{A+V}{I} - \left(c_1 \pi_6 \frac{D}{N} + c_5 \theta_1 \frac{T}{P+I} + c_4 \theta_2 \frac{B}{P+I} + c_6 \theta_3 \frac{K}{P+I} + \gamma_5 + \mu + \eta\right) P\right] \\
+ \quad & \lambda_{11} \left[c_6 \left(\alpha_3 C \frac{K}{C+I} + \omega_5 R \frac{K}{R+I} + \omega_6 A \frac{K}{I} + \omega_7 B \frac{K}{I} + \theta_3 P \frac{K}{P+I}\right) - \left(c_1 \pi_7 \frac{D}{N} + \tau_4 \frac{A+V}{I} + \gamma_{10} + \mu + \zeta_6 \frac{A+V}{I}\right) K\right]
\end{aligned} \tag{5.7}$$

where $\lambda_i, i = 1, 2, ..., 11$, represent the adjoint variables associated with the state variables of the model (5.2).

We introduce the Lagrangian $\mathcal{L}$ associated with the problem (5.2), which corresponds to the Hamiltonian augmented by the penalty terms and is defined by

$$\begin{aligned}\mathcal{L} \quad = \quad & \mathcal{H} - p_{11}u_1 - p_{12}(1-u_1) - p_{21}u_2 - p_{22}(1-u_2) - p_{31}u_3 - p_{32}(1-u_3) \\ & -p_{41}u_4 - p_{42}(1-u_4) - p_{51}u_5 - p_{52}(1-u_5) - p_{61}u_6 - p_{62}(1-u_6)\end{aligned}$$

where $p_{ij}(t) \geq 0$ are penalty coefficients verifying :

$$\left\{\begin{array}{lcl} p_{11}u_1 & = & p_{12}(1-u_1)=0 \ ; \ p_{21}u_2 = p_{22}(1-u_2)=0 \ ; \ p_{31}u_3 = p_{32}(1-u_3)=0 \ ; \\ p_{41}u_4 & = & p_{42}(1-u_4)=0 \ ; \ p_{51}u_5 = p_{52}(1-u_5)=0 \ ; \ p_{61}u_6 - p_{62}(1-u_6)=0 \ . \end{array}\right. \tag{5.8}$$

The standard existence result for minimizing control problem as appeared in [15] is adapted as follows.

Theorem 5.1. ***(Existence and well-posedness of the control problem)***
There exists a sixfold optimal control $\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*\right) \in \mathcal{U}$ *satisfying* (5.5) *subject to the control system* (5.2) *with the initial conditions* (2.2).

Proof : The existence of the optimal control is obtained thanks to a result of Fleming and Rishel in [15]. Thanks to a result of Lukes [20] which ensures the existence of solutions for system (5.2), the set of controls and corresponding solutions is non-empty. In addition the set of controls $\mathcal{U}$ is a closed convex by definition and the vector field of system (5.2) is bounded. Also the integrand of the objective function is clearly convex and $g(t, X, c)$ in (5.3) is convex with respect to c. On the other hand there exist $a_1, a_2 > 0$ and $\beta > 1$ such that

$$\begin{aligned} & w_1S + w_2E + w_3F + w_4B + w_5T + w_6P + w_{12}K + \frac{1}{2}\Big[w_7u_1^2(t) + w_8u_2^2(t) + w_9u_3^2(t) + w_{10}u_4^2(t) + w_{11}u_5^2(t)) + w_{13}u_6^2(t)\Big] \\ & \qquad \geq a_1\left(|u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 + |u_5|^2 + |u_6|^2\right)^{\frac{\beta}{2}} - a_2 \end{aligned}$$

since the state variables are bounded.
Then, we deduce the existence of an optimal control $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*)$ that minimizes the objective function $J(u_1, u_2, u_3, u_4, u_5, u_6)$. □

For more details, see [18]. This article provides further explanations.

Theorem 5.2. *Let* $\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*\right)$ *be a given optimal control, and let* $(C, R, A, V, S, E, F, B, K, T)$ *be the solution of the corresponding state system* (5.2). *Then there exist adjoint variables* $\lambda_i, i =$

$1, ..., 11$, *satisfying :*

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1-\lambda_5)c_1\pi_1\frac{D(N-C)}{N^2}+(\lambda_1-\lambda_4)\alpha_1\frac{(T+B)I}{(C+I)^2}+(\lambda_1-\lambda_8)c_4\alpha_2\frac{BI}{(C+I)^2}+(\lambda_1-\lambda_3)\sigma_2+\lambda_1\mu\\
&+(\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2}+(\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2}+(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2}+(\lambda_5-\lambda_8)c_1\pi_4\frac{DB}{N^2}+(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2}\\
&+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}+(\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2}\\
&+(\lambda_1-\lambda_{11})c_6\alpha_3\frac{KI}{(C+I)^2};\\
\frac{d\lambda_2}{dt} &= (\lambda_2-\lambda_1)\gamma_6+(\lambda_2-\lambda_1)c_1\pi_1\frac{DC}{N^2}+(\lambda_2-\lambda_1)c_1\pi_5\frac{D(N-R)}{N^2}+(\lambda_2-\lambda_6)c_5\omega_3\frac{TI}{(R+I)^2}+\lambda_2\mu\\
&+(\lambda_2-\lambda_8)c_4\omega_4\frac{BI}{(R+I)^2}+(\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2}+(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2}+(\lambda_5-\lambda_8)c_1\pi_4\frac{DB}{N^2}\\
&+(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2}+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}\\
&+(\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2}+(\lambda_2-\lambda_{11})c_6\alpha_3\frac{KI}{(C+I)^2};\\
\frac{d\lambda_3}{dt} &= (\lambda_3-\lambda_1)\gamma_4+(\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2}+(\lambda_4-\lambda_1)\alpha_1\frac{(T+B)C}{(C+I)^2}+(\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2}\\
&+(\lambda_3-\lambda_2)\nu_2+(\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2}+(\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2}+(\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2}+(\lambda_3-\lambda_5)c_1\pi_3\frac{D(N-A)}{N^2}\\
&+(\lambda_3-\lambda_8)c_4\nu_3\frac{B(V+T+B+K)}{I^2}+(\lambda_3-\lambda_9)c_5\omega_2\frac{T(V+T+B+K)}{I^2}+\lambda_3\zeta_1\frac{(T+B)(V+T+B+K)}{I^2}+\lambda_3\mu\\
&(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2}-\lambda_4\zeta_2\frac{(T+B+K)V}{I^2}+(\lambda_5-\lambda_8)c_1\pi_4\frac{DB}{N^2}+(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2}+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}\\
&+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}+(\lambda_7-\lambda_{10})\tau_1\frac{F(F+T+B+K)}{(F+I)^2}+(\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2}\\
&+\lambda_7\zeta_3\frac{F(F+T+B+K)}{(F+I)^2}+(\lambda_8-\lambda_{10})\tau_2\frac{B(T+B+K)}{I^2}+(\lambda_9-\lambda_8)c_5\omega_1\frac{TB}{I^2}+\lambda_8\zeta_4\frac{B(T+B+K)}{I^2}\\
&+(\lambda_9-\lambda_{10})\tau_3\frac{T(T+B+K)}{I^2}+\lambda_9\zeta_5\frac{T(T+B+K)}{I^2}++(\lambda_{11}-\lambda_1)c_6\alpha_3\frac{KC}{(C+I)^2}+(\lambda_{11}-\lambda_2)c_6\omega_5\frac{KR}{(R+I)^2}\\
&+(\lambda_3-\lambda_{11})c_6\omega_6\frac{K(V+T+B+K)}{I^2}+(\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2}+(\lambda_{11}-\lambda_{10})c_6\theta_3\frac{PK}{(P+I)^2}\\
&+(\lambda_{11}-\lambda_{10})\tau_4\frac{B(T+B+K)}{I^2}+\lambda_{11}\zeta_4\frac{K(T+B+K)}{I^2}+(\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2};
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
\frac{d\lambda_4}{dt} \quad = \quad & (\lambda_3-\lambda_1)\gamma_9+(\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2}+(\lambda_4-\lambda_1)\alpha_1\frac{(T+B)C}{(C+I)^2}+(\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2}+(\lambda_4-\lambda_3)\sigma_1\\
&+(\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2}+(\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2}+(\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2}+(\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2}\\
&+(\lambda_8-\lambda_3)c_4\nu_3\frac{BA}{I^2}+(\lambda_9-\lambda_3)c_5\omega_2\frac{TA}{I^2}-\lambda_3\zeta_1\frac{(T+B+K)A}{I^2}+\lambda_4\mu+(\lambda_4-\lambda_5)c_1\pi_2\frac{D(N-V)}{N^2}\\
&+\lambda_4\zeta_2\frac{(T+B)(A+T+B+K)}{I^2}+(\lambda_5-\lambda_8)c_1\pi_4\frac{DB}{N^2}+(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2}+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}\\
&+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}+(\lambda_7-\lambda_{10})\tau_1\frac{F(F+T+B)}{(F+I)^2}+(\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2}\\
&+\lambda_7\zeta_3\frac{F(F+T+B)}{(F+I)^2}+(\lambda_8-\lambda_{10})\tau_2\frac{B(T+B+K)}{I^2}+(\lambda_9-\lambda_8)c_5\omega_1\frac{TB}{I^2}+\lambda_8\zeta_4\frac{B(T+B+K)}{I^2}\\
&+(\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2}+(\lambda_9-\lambda_{10})\tau_3\frac{T(T+B+K)}{I^2}+\lambda_9\zeta_5\frac{T(T+B+K)}{I^2}++(\lambda_{11}-\lambda_1)c_6\alpha_3\frac{KC}{(C+I)^2}\\
&+(\lambda_{11}-\lambda_2)c_6\omega_5\frac{KR}{(R+I)^2}+(\lambda_3-\lambda_{11})c_6\omega_6\frac{K(V+T+B+K)}{I^2}+(\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2}+(\lambda_{11}-\lambda_{10})c_6\theta_3\frac{PK}{(P+I)^2}\\
&+(\lambda_{11}-\lambda_{10})\tau_4\frac{B(T+B+K)}{I^2}+\lambda_{11}\zeta_4\frac{K(T+B+K)}{I^2};
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_5}{dt} \quad = \quad & -w_1+(\lambda_5-\lambda_1)\gamma_1+(\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2}+(\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2}+(\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2}\\
&+(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2}+(\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2}+(\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-D)}{N^2}+\lambda_5\mu\\
&+(\lambda_5-\lambda_6)c_2\beta_2\frac{(E+F+T)(N-S)}{N^2}+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\pi_4\frac{TF}{N^2}+(\lambda_{11}-\lambda_5)c_1\pi_7\frac{K(N-D)}{N^2};
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_6}{dt} \quad = \quad & -w_2+(\lambda_6-\lambda_1)\gamma_2+(\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2}+(\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2}+(\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2}\\
&+(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2}+(\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2}+(\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-D)}{N^2}+\lambda_6\mu+(\lambda_9-\lambda_7)c_5\pi_4\frac{TF}{N^2}\\
&+(\lambda_5-\lambda_6)c_2\beta_2\frac{S\Big(N-(E+F+T)\Big)}{N^2}+(\lambda_6-\lambda_7)c_3\beta_3\frac{(F+T)(N-E)}{N^2}+(\lambda_{11}-\lambda_5)c_1\pi_7\frac{K(N-D)}{N^2};
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_7}{dt} \quad = \quad & -w_3+(\lambda_7-\lambda_1)\gamma_2+(\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2}+(\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2}+(\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2}\\
&+(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2}+(\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2}+(\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-D)}{N^2}+\lambda_7\mu\\
&+(\lambda_5-\lambda_6)c_2\beta_2\frac{S\Big(N-(E+F+T)\Big)}{N^2}+(\lambda_6-\lambda_7)c_3\beta_3\frac{(F+T)(N-E)}{N^2}+(\lambda_7-\lambda_9)c_5\beta_4\frac{T(N-F)}{N^2}\\
&+(\lambda_7-\lambda_{10})\tau_1\frac{(A+V)I}{(F+I)^2}+\lambda_7\zeta_3\frac{(A+V)I}{(F+I)^2}+(\lambda_{11}-\lambda_5)c_1\pi_7\frac{K(N-D)}{N^2};
\end{aligned}
$$

$$\begin{aligned}
\frac{d\lambda_8}{dt} \quad = \quad & -w_4 + (\lambda_8-\lambda_1)\gamma_7 + (\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2} + (\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2} + (\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2} + (\lambda_8-\lambda_3)\nu_1 \\
& +(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2} + (\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2} + (\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2} + (\lambda_1-\lambda_4)\alpha_1\frac{C(C+A+V+K)}{(C+I)^2} \\
& +(\lambda_1-\lambda_8)c_4\alpha_2\frac{C(C+A+V+T+K)}{(C+I)^2} + (\lambda_2-\lambda_8)c_4\omega_4\frac{R(R+A+V+T+K)}{(R+I)^2} + \lambda_8\mu + (\lambda_3-\lambda_9)c_5\omega_2\frac{TA}{I^2} \\
& +(\lambda_3-\lambda_8)c_4\nu_3\frac{A(A+V+T+K)}{I^2} + \lambda_3\zeta_1\frac{A(A+V)}{I^2} + \lambda_4\zeta_2\frac{V(A+V)}{I^2} + (\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2} \\
& +(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2} + (\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2} + (\lambda_{10}-\lambda_7)\tau_1\frac{F(A+V)}{(F+I)^2} - \lambda_7\zeta_3\frac{F(A+V)}{(F+I)^2} \\
& +(\lambda_8-\lambda_{10})\tau_2\frac{(A+V)(A+V+T+K)}{I^2} + \lambda_8\zeta_4\frac{(A+V)(A+V+T+K)}{I^2} + (\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2} \\
& +(\lambda_{10}-\lambda_9)\tau_3\frac{T(A+V)}{I^2} - \lambda_9\zeta_5\frac{T(A+V)}{I^2} + (\lambda_8-\lambda_5)c_1\pi_4\frac{D(N-B)}{N^2} + (\lambda_8-\lambda_9)c_5\omega_1\frac{T(A+V+T+K)}{I^2} \\
& +(\lambda_8-\lambda_{10})c_4\theta_2\frac{P(P+A+V+T+K)}{(P+I)^2} + (\lambda_{11}-\lambda_2)c_6\omega_5\frac{KR}{(R+I)^2} + (\lambda_3-\lambda_{11})c_6\omega_6\frac{KA}{I^2} \\
& +(\lambda_8-\lambda_{11})c_6\omega_7\frac{K(A+V+T+K)}{I^2} + (\lambda_{11}-\lambda_{10})c_6\theta_3\frac{KP}{(P+I)^2} + (\lambda_{10}-\lambda_{11})\tau_4\frac{K(A+V)}{I^2} \\
& +(\lambda_{11}-\lambda_1)c_6\alpha_3\frac{KC}{(C+I)^2} - \lambda_{11}\zeta_6\frac{K(A+V)}{I^2} + (\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2};
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_9}{dt} \quad = \quad & -w_5 + (\lambda_9-\lambda_1)\gamma_8 + (\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2} + (\lambda_1-\lambda_4)\alpha_1\frac{C(C+A+V)}{(C+I)^2} + (\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2} \\
& +(\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2} + (\lambda_2-\lambda_9)c_5\omega_3\frac{R(R+A+V+B+K)}{(R+I)^2} + (\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2} \\
& +(\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2} + (\lambda_8-\lambda_3)c_4\nu_3\frac{BA}{I^2} + (\lambda_3-\lambda_9)c_5\omega_2\frac{A(A+V+B+K)}{I^2} + \lambda_3\zeta_1\frac{A(A+V)}{I^2} \\
& +(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2} + \lambda_4\zeta_2\frac{V(A+V)}{I^2} + (\lambda_5-\lambda_6)c_2\beta_2\frac{S\Big(N-(E+F+T)\Big)}{N^2} + (\lambda_6-\lambda_7)c_3\beta_3\frac{E\Big(N-(F+T)\Big)}{N^2} \\
& +(\lambda_{10}-\lambda_7)\tau_1\frac{F(A+V)}{(F+I)^2} + (\lambda_7-\lambda_{10})c_5\beta_4\frac{F(N-T)}{N^2} - \lambda_7\zeta_3\frac{F(A+V)}{(F+I)^2} + (\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2} \\
& +(\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2} + (\lambda_8-\lambda_9)c_5\omega_1\frac{B(A+V+B+K)}{I^2} - \lambda_8\zeta_4\frac{B(A+V)}{I^2} + (\lambda_{10}-\lambda_8)\tau_2\frac{B(A+V)}{I^2} \\
& +(\lambda_{10}-\lambda_9)c_5\theta_1\frac{P(P+A+V+B+K)}{(P+I)^2} + (\lambda_9-\lambda_{10})\tau_3\frac{(A+V)(A+V+B+K)}{I^2} + (\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-B)}{N^2} \\
& +\lambda_9\mu + \lambda_9\zeta_5\frac{(A+V)(A+V+B+K)}{I^2} + (\lambda_{11}-\lambda_2)c_6\omega_5\frac{KR}{(R+I)^2} + (\lambda_{11}-\lambda_3)c_6\omega_6\frac{KA}{I^2} + (\lambda_{11}-\lambda_8)c_6\omega_7\frac{KB}{I^2} \\
& +(\lambda_{11}-\lambda_{10})c_6\theta_3\frac{KP}{(P+I)^2} + (\lambda_{10}-\lambda_{11})\tau_4\frac{K(A+V)}{I^2} - \lambda_{11}\zeta_6\frac{K(A+V)}{I^2} + (\lambda_{11}-\lambda_1)c_6\alpha_3\frac{KC}{(C+I)^2} \\
& +(\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{d\lambda_{10}}{dt} &= -w_6+(\lambda_{10}-\lambda_1)\gamma_5+(\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2}+(\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2}+(\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2}+(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2}+(\lambda_5-\lambda_8)c_1\pi_4\\
&+(\lambda_{10}-\lambda_5)c_1\pi_6\frac{D(N-P)}{N^2}+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}\\
&+(\lambda_{10}-\lambda_8)c_4\theta_2\frac{BI}{(P+I)^2}+(\lambda_{10}-\lambda_9)c_5\theta_1\frac{TI}{(P+I)^2}+\lambda_{10}\mu+\lambda_{10}\eta+(\lambda_5-\lambda_{11})c_1\pi_7\frac{DK}{N^2}+(\lambda_{10}-\lambda_{11})c_6\theta_3\frac{KI}{(P+I)^2};
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_{11}}{dt} &= -w_{12}+(\lambda_{11}-\lambda_1)\gamma_{10}+(\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2}+(\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2}+(\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2}+(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2}\\
&+(\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2}+(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2}+(\lambda_4-\lambda_1)\alpha_1\frac{C(T+B)}{(C+I)^2}+(\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2}+(\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2}\\
&+\lambda_{11}\mu+(\lambda_9-\lambda_3)c_5\omega_2\frac{TA}{I^2}+(\lambda_8-\lambda_3)c_4\nu_3\frac{BA}{I^2}+\lambda_3\zeta_1\frac{A(A+V)}{I^2}+\lambda_4\zeta_2\frac{V(A+V)}{I^2}+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}\\
&+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2}+(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}+(\lambda_{10}-\lambda_7)\tau_1\frac{F(A+V)}{(F+I)^2}-\lambda_7\zeta_3\frac{F(A+V)}{(F+I)^2}+(\lambda_8-\lambda_{10})\tau_2\frac{B(A+V)}{I^2}\\
&-\lambda_8\zeta_4\frac{B(A+V)}{I^2}+(\lambda_9-\lambda_8)c_5\omega_1\frac{TB}{I^2}+(\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2}+(\lambda_{10}-\lambda_9)\tau_3\frac{T(A+V)}{I^2}-\lambda_9\zeta_5\frac{T(A+V)}{I^2}+(\lambda_5-\lambda_8)c_1\pi\\
&+(\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2}+(\lambda_1-\lambda_{11})c_6\alpha_3\frac{C(C+A+V+T+B)}{(C+I)^2}+(\lambda_2-\lambda_{11})c_6\omega_5\frac{R(R+A+V+T+B)}{(R+I)^2}\\
&+(\lambda_3-\lambda_{11})c_6\omega_6\frac{A(A+V+T+B)}{I^2}+(\lambda_{11}-\lambda_5)c_1\pi_7\frac{D(N-K)}{N^2}+(\lambda_8-\lambda_{11})c_6\omega_7\frac{B(A+V+T+B)}{I^2}\\
&+(\lambda_{11}-\lambda_{10})\tau_4\frac{(A+V)(A+V+T+B)}{I^2}+(\lambda_{10}-\lambda_{11})c_6\theta_3\frac{P(P+A+V+T+B)}{(P+I)^2}+\lambda_{11}\zeta_6\frac{(A+V)(A+V+T+B)}{I^2};
\end{aligned}
$$

provided with transversality condition

$$\lambda_i(T_f)=0,\quad i=1,2,...,11.$$

The optimal controls $\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*\right)$ *are then represented as follows :*

$$\left\{\begin{array}{lcl} u_1^* & = & \max\left\{0, \min\left\{1, \dfrac{\Big((\lambda_5-\lambda_1)\pi_1 C+(\lambda_5-\lambda_2)\pi_5 R+(\lambda_5-\lambda_3)\pi_3 A+(\lambda_5-\lambda_4)\pi_2 V+(\lambda_5-\lambda_8)\pi_4 B+(\lambda_5-\lambda_{10})\pi_6 P+(\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N}\right\}\right\} \\ u_2^* & = & \max\left\{0, \min\left\{1, \dfrac{(\lambda_5-\lambda_6)\beta_2 S(E+F+T)}{w_8 N}\right\}\right\} \\ u_3^* & = & \max\left\{0, \min\left\{1, \dfrac{(\lambda_6-\lambda_7)\beta_3 E(F+T)}{w_9 N}\right\}\right\} \\ u_4^* & = & \max\left\{0, \min\left\{1, \dfrac{(\lambda_8-\lambda_1)\alpha_2 \dfrac{BC}{C+I}+(\lambda_8-\lambda_2)\omega_4 \dfrac{BR}{R+I}+(\lambda_8-\lambda_3)\nu_3 \dfrac{BA}{I}+(\lambda_8-\lambda_{10})\theta_2 \dfrac{BP}{P+I}}{w_{10}}\right\}\right\} \\ u_5^* & = & \max\left\{0, \min\left\{1, \dfrac{(\lambda_9-\lambda_2)\omega_3 \dfrac{TR}{R+I}+(\lambda_9-\lambda_3)\omega_2 \dfrac{TA}{I}+(\lambda_9-\lambda_7)\beta_4 \dfrac{TF}{N}+(\lambda_9-\lambda_8)\omega_1 \dfrac{TB}{I}+(\lambda_9-\lambda_{10})\theta_1 \dfrac{TP}{P+I}}{w_{11}}\right\}\right\} \\ u_6^* & = & \max\left\{0, \min\left\{1, \dfrac{(\lambda_{11}-\lambda_1)\alpha_3 \dfrac{KC}{C+I}+(\lambda_{11}-\lambda_2)\omega_5 \dfrac{KR}{R+I}+(\lambda_{11}-\lambda_3)\omega_6 \dfrac{KA}{I}+(\lambda_{11}-\lambda_8)\omega_7 \dfrac{KB}{I}+(\lambda_{11}-\lambda_{10})\theta_3 \dfrac{KP}{P+I}}{w_{13}}\right\}\right\}. \end{array}\right. \tag{5.10}$$

Proof : As mentioned earlier, the characterization of the optimal solution is obtained by applying the Pontryagin's maximum principle. The system of ordinary differential equations (5.9) governing the adjoint variables is derived by differentiating the Hamiltonian.

To obtain the optimal control formulation expressed by (5.10), we solve the constraint equation obtained by taking the derivative of the Lagrangian $\mathcal{L}$ with respect to $\left(u_1, u_2, u_3, u_4, u_5, u_6\right)$.

$$\begin{array}{lcl} \dfrac{\partial \mathcal{L}}{\partial u_1} & = & w_7 u_1 - \dfrac{\Big((\lambda_5-\lambda_1)\pi_1 C+(\lambda_5-\lambda_2)\pi_5 R+(\lambda_5-\lambda_3)\pi_3 A+(\lambda_5-\lambda_4)\pi_2 V+(\lambda_5-\lambda_8)\pi_4 B+(\lambda_5-\lambda_{10})\pi_6 P+(\lambda_5-\lambda_{11})\pi_7 K\Big)D}{N} \\ & & -p_{11}+p_{12}=0 \quad ; \\ \dfrac{\partial \mathcal{L}}{\partial u_2} & = & w_8 u_2 - \dfrac{(\lambda_5-\lambda_6)\beta_2 S(E+F+T)}{N} - p_{21}+p_{22}=0 \quad ; \\ \dfrac{\partial \mathcal{L}}{\partial u_3} & = & w_9 u_3 - \dfrac{(\lambda_6-\lambda_7)\beta_3 E(F+T)}{N} - p_{31}+p_{32}=0 \quad ; \\ \dfrac{\partial \mathcal{L}}{\partial u_4} & = & w_{10} u_4 - \left[(\lambda_8-\lambda_1)\alpha_2 \dfrac{BC}{C+I}+(\lambda_8-\lambda_2)\omega_4 \dfrac{BR}{R+I}+(\lambda_8-\lambda_3)\nu_3 \dfrac{BA}{I}+(\lambda_8-\lambda_{10})\theta_2 \dfrac{BP}{P+I}\right] - p_{41}+p_{42}=0 \quad ; \\ \dfrac{\partial \mathcal{L}}{\partial u_5} & = & w_{11} u_5 - \left[(\lambda_9-\lambda_2)\omega_3 \dfrac{TR}{R+I}+(\lambda_9-\lambda_3)\omega_2 \dfrac{TA}{I}+(\lambda_9-\lambda_7)\beta_4 \dfrac{TF}{N}+(\lambda_9-\lambda_8)\omega_1 \dfrac{TB}{I}+(\lambda_9-\lambda_{10})\theta_1 \dfrac{TP}{P+I}\right] - p_{51}+p_{52}=0 \quad ; \\ \dfrac{\partial \mathcal{L}}{\partial u_6} & = & w_{13} u_6 - \left[(\lambda_{11}-\lambda_1)\alpha_3 \dfrac{KC}{C+I}+(\lambda_{11}-\lambda_2)\omega_5 \dfrac{KR}{R+I}+(\lambda_{11}-\lambda_3)\omega_6 \dfrac{KA}{I}+(\lambda_{11}-\lambda_8)\omega_7 \dfrac{KB}{I}+(\lambda_{11}-\lambda_{10})\theta_3 \dfrac{KP}{P+I}\right] - p_{61}+p_{62} \end{array}$$

After solving, we obtain

$$u_1^* = \frac{\Big((\lambda_5-\lambda_1)\pi_1 C+(\lambda_5-\lambda_2)\pi_5 R+(\lambda_5-\lambda_3)\pi_3 A+(\lambda_5-\lambda_4)\pi_2 V+(\lambda_5-\lambda_8)\pi_4 B+(\lambda_5-\lambda_{10})\pi_6 P+(\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N} + \frac{1}{w_7}\Big(p_{11}-p_{12}\Big)$$

$$u_2^* = \frac{(\lambda_5-\lambda_6)\beta_2 S(E+F+T)}{w_8 N} + \frac{1}{w_8}\Big(p_{21}-p_{22}\Big)$$

$$u_3^* = \frac{(\lambda_6-\lambda_7)\beta_3 E(F+T)}{w_9 N} + \frac{1}{w_9}\Big(p_{31}-p_{32}\Big)$$

$$u_4^* = \frac{(\lambda_8-\lambda_1)\alpha_2\dfrac{BC}{C+I}+(\lambda_8-\lambda_2)\omega_4\dfrac{BR}{R+I}+(\lambda_8-\lambda_3)\nu_3\dfrac{BA}{I}+(\lambda_8-\lambda_{10})\theta_2\dfrac{BP}{P+I}}{w_{10}} + \frac{1}{w_{10}}\Big(p_{41}-p_{42}\Big)$$

$$u_5^* = \frac{(\lambda_9-\lambda_2)\omega_3\dfrac{TR}{R+I}+(\lambda_9-\lambda_3)\omega_2\dfrac{TA}{I}+(\lambda_9-\lambda_7)\beta_4\dfrac{TF}{N}+(\lambda_9-\lambda_8)\omega_1\dfrac{TB}{I}+(\lambda_9-\lambda_{10})\theta_1\dfrac{TP}{P+I}}{w_{11}} + \frac{1}{w_{11}}\Big(p_{51}-p_{52}\Big)$$

$$u_6^* = \frac{(\lambda_{11}-\lambda_1)\alpha_3\dfrac{KC}{C+I}+(\lambda_{11}-\lambda_2)\omega_5\dfrac{KR}{R+I}+(\lambda_{11}-\lambda_3)\omega_6\dfrac{KA}{I}+(\lambda_{11}-\lambda_8)\omega_7\dfrac{KB}{I}+(\lambda_{11}-\lambda_{10})\theta_3\dfrac{KP}{P+I}}{w_{13}} + \frac{1}{w_{13}}\Big(p_{61}-p_{62}\Big)$$

To obtain the explicit formula for optimal control without $p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}, p_{41}, p_{42}, p_{51}, p_{52}, p_{61}$, and p_{62}, we use standard techniques. Three specific cases are examined.

$(*)$ Let $\Re_1 = \Big\{t \in \mathbb{R}^+ / 0 < u_1^* < 1\Big\}$.

For all $t \in \Re_1$, we have $p_{11}(t)u_1^*(t) = p_{12}(t)(1-u_1^*(t)) = 0$ leads to $p_{11}(t) = p_{12}(t) = 0$.

So the optimal control is :

$$u_1^* = \frac{\Big((\lambda_5-\lambda_1)\pi_1 C+(\lambda_5-\lambda_2)\pi_5 R+(\lambda_5-\lambda_3)\pi_3 A+(\lambda_5-\lambda_4)\pi_2 V+(\lambda_5-\lambda_8)\pi_4 B+(\lambda_5-\lambda_{10})\pi_6 P+(\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N}.$$

$(**)$ Let $\Re_2 = \Big\{t \in \mathbb{R}^+ / u_1^* = 1\Big\}$.

For all $t \in \Re_2$, we have $p_{11}(t)u_1^*(t) = p_{12}(t)(1-u_1^*(t)) = 0$ leads to $p_{11}(t) = 0$.

Then the optimal control is :

$$u_1^* = \frac{\Big((\lambda_5-\lambda_1)\pi_1 C+(\lambda_5-\lambda_2)\pi_5 R+(\lambda_5-\lambda_3)\pi_3 A+(\lambda_5-\lambda_4)\pi_2 V+(\lambda_5-\lambda_8)\pi_4 B+(\lambda_5-\lambda_{10})\pi_6 P+(\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N} - \frac{p_{12}}{w_7} =$$

Thus

$$u_1^* = \frac{\Big((\lambda_5-\lambda_1)\pi_1 C + (\lambda_5-\lambda_2)\pi_5 R + (\lambda_5-\lambda_3)\pi_3 A + (\lambda_5-\lambda_4)\pi_2 V + (\lambda_5-\lambda_8)\pi_4 B + (\lambda_5-\lambda_{10})\pi_6 P + (\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N} \geq 1,$$

since $\dfrac{p_{12}}{w_7} \geq 0$ given that $p_{12}(t) \geq 0$ and $w_7 > 0$.

$(***)$ Let $\Re_3 = \Big\{t \in \mathbb{R}^+ / u_1^* = 0\Big\}$.

For all $t \in \Re_3$, we have $p_{11}(t)u_1^*(t) = p_{12}(t)(1-u_1^*(t)) = 0$ leads to $p_{12}(t) = 0$.

Then the optimal control is :

$$u_1^* = \frac{\Big((\lambda_5-\lambda_1)\pi_1 C + (\lambda_5-\lambda_2)\pi_5 R + (\lambda_5-\lambda_3)\pi_3 A + (\lambda_5-\lambda_4)\pi_2 V + (\lambda_5-\lambda_8)\pi_4 B + (\lambda_5-\lambda_{10})\pi_6 P + (\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N} + \frac{p_{11}}{w_7} = 0.$$

Thus

$$u_1^* = \frac{\Big((\lambda_5-\lambda_1)\pi_1 C + (\lambda_5-\lambda_2)\pi_5 R + (\lambda_5-\lambda_3)\pi_3 A + (\lambda_5-\lambda_4)\pi_2 V + (\lambda_5-\lambda_8)\pi_4 B + (\lambda_5-\lambda_{10})\pi_6 P + (\lambda_5-\lambda_{11})\pi_7 K\Big)D}{w_7 N} \leq 0$$

since $\dfrac{p_{11}}{w_7} \geq 0$ given that $p_{11}(t) \geq 0$ and $w_7 > 0$.

From $(*)$, $(**)$ and $(***)$, we conclude that the optimal control u_1^* is rewritten as follows

$$u_1^* = \begin{cases} 0 \text{ if } & r_1^* \leq 0 \\ r_1^* \text{ if } & 0 < r_1^* < 1 \\ 1 \text{ if } & r_1^* \geq 1 \end{cases}$$

Analogously, we show that :

$$u_2^* = \begin{cases} 0 \text{ if } & r_2^* \leq 0 \\ r_2^* \text{ if } & 0 < r_2^* < 1 \\ 1 \text{ if } & r_2^* \geq 1 \end{cases}$$

$$u_3^* = \begin{cases} 0 \text{ if } & r_3^* \leq 0 \\ r_3^* \text{ if } & 0 < r_3^* < 1 \\ 1 \text{ if } & r_3^* \geq 1 \end{cases}$$

$$u_4^* = \begin{cases} 0 \text{ if } & r_4^* \leq 0 \\ r_4^* \text{ if } & 0 < r_1^* < 1 \\ 1 \text{ if } & r_4^* \geq 1 \end{cases}$$

$$u_5^* = \begin{cases} 0 \text{ if } & r_5^* \leq 0 \\ r_5^* \text{ if } & 0 < r_5^* < 1 \\ 1 \text{ if } & r_5^* \geq 1 \end{cases}$$

$$u_6^* = \begin{cases} 0 \text{ if } & r_6^* \leq 0 \\ r_6^* \text{ if } & 0 < r_6^* < 1 \\ 1 \text{ if } & r_6^* \geq 1 \end{cases}$$

where,

$$\left\{\begin{array}{lcl}
r_2^* & = & \dfrac{(\lambda_5-\lambda_6)\beta_2 S(E+F+T)}{w_8 N} \\[2ex]
r_3^* & = & \dfrac{(\lambda_6-\lambda_7)\beta_3 E(F+T)}{w_9 N} \\[2ex]
r_4^* & = & \dfrac{(\lambda_8-\lambda_1)\alpha_2\dfrac{BC}{C+I}+(\lambda_8-\lambda_2)\omega_4\dfrac{BR}{R+I}+(\lambda_8-\lambda_3)\nu_3\dfrac{BA}{I}+(\lambda_8-\lambda_{10})\theta_2\dfrac{BP}{P+I}}{w_{10}} \\[2ex]
r_5^* & = & \dfrac{(\lambda_9-\lambda_2)\omega_3\dfrac{TR}{R+I}+(\lambda_9-\lambda_3)\omega_2\dfrac{TA}{I}+(\lambda_9-\lambda_7)\beta_4\dfrac{TF}{N}+(\lambda_9-\lambda_8)\omega_1\dfrac{TB}{I}+(\lambda_9-\lambda_{10})\theta_1\dfrac{TP}{P+I}}{w_{11}} \\[2ex]
r_6^* & = & \dfrac{(\lambda_{11}-\lambda_1)\alpha_3\dfrac{KC}{C+I}+(\lambda_{11}-\lambda_2)\omega_5\dfrac{KR}{R+I}+(\lambda_{11}-\lambda_3)\omega_6\dfrac{KA}{I}+(\lambda_{11}-\lambda_8)\omega_7\dfrac{KB}{I}+(\lambda_{11}-\lambda_{10})\theta_3\dfrac{KP}{P+I}}{w_{13}}
\end{array}\right.$$

This ends the proof. □

6 Numerical simulations of the control system

To illustrate the impact of counterradicalization and counterterrorism strategies on the dynamic evolution of the core sub-population of violent extremists, we have carried out a detailed numerical simulation. For ease of reading, we present the evolution of the different population classes (S, E, F, B, K, T) without control, represented in red, and with control, represented in blue. We highlight the scenario in which the different control strategies prove most effective. The results, obtained with a MATLAB implementation using an explicit Euler scheme, are based on the parameter values described in Table 2 and the following initial conditions : C(0)=150000, R(0)=8, A(0)=150, V(0)=150, S(0)=25000, E(0)=1500, F(0)=400, B(0)=100, T(0)=150, K(0)=150, P(0)=20.

Table 2 – Parameters values estimated

Parameters	Threat Persistence Values
Λ	22500
γ_1	0.00046
γ_2	0.000028
γ_3	0.000000111
γ_4	0.12
γ_5	0.0000016
γ_6	0.0026
γ_7	0.002
γ_8	0.0000011
γ_9	0.011
γ_{10}	0.000011
π_1	1
π_2	0.00534
π_3	0.02
π_4	0.0014
π_5	0.04
π_6	0.5
π_7	0.1
θ_1	0.0032
θ_2	0.0032
θ_3	0.00032
η	0.15
ζ_1	0.17
ζ_2	0.17
ζ_3	0.07
ζ_4	0.2
ζ_5	0.1
ζ_6	0.24
μ	0.08
ν_1	0.002
ν_2	0.002
ν_3	0.01
τ_1	0.00002
τ_2	0.045
τ_3	0.00045
τ_4	0.00045
β_2	0.75
β_3	0.82
β_4	0.98
σ_1	0.11
σ_2	0.01
α_1	0.02
α_2	4
α_3	2
ω_1	0.29
ω_2	0.4
ω_3	1.1
ω_4	1
ω_5	1
ω_6	0.42
ω_7	0.002

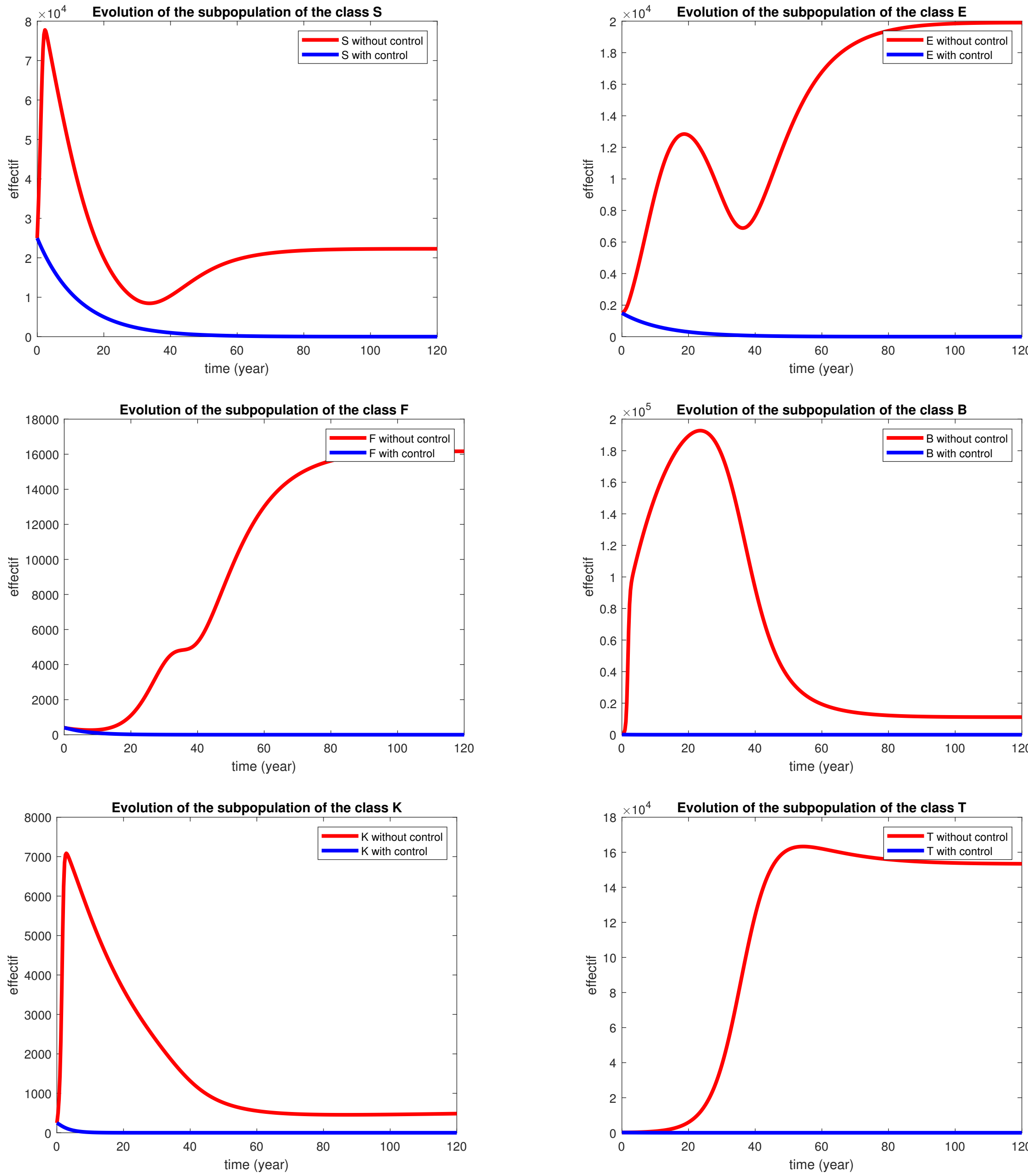

FIGURE 2 – Dynamics of individuals in the S, E, F, B, K, and T classes without control and with control $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$.

An analysis of figure 2 reveals a significant trend : a marked reduction in terrorist threats, fanatical ideology, banditry and drug trafficking. These observations underline the potential effectiveness of the various strategies to combat violent extremism and narcoterrorism when properly implemented. This reduction in threats suggests that concerted efforts, both preventive and repressive, can lead to a rapid stabilization of the security climate in the Sahel region.

A closer look at the trends observed in Figure 2 leads to several important conclusions. Firstly, the decline in terrorist threats and fanatical ideology indicates the effectiveness of preventive measures such as awareness campaigns, the promotion of social cohesion and the fight against radicalization. Similarly, the decline in banditry and drug trafficking suggests the positive impact of security operations aimed at disrupting criminal networks and reducing their influence in the region.

These results underline the importance of an integrated and equilibrium approach in the fight against fanatical insurgency and narcoterrorism in the Sahel. They highlight the need for close coordination between preventive and repressive measures, as well as the relevance of a multidimensional approach to tackling the root causes of these threats. They also underline the importance of regional and international cooperation in designing and implementing effective control strategies.

7 Conclusion

This study of optimal multi-objective control of fanatical insurgency and narcoterrorism in the Sahel highlights the crucial importance of preventive measures and effective security operations in achieving overall stability. It underscores the need for equilibrium between preventive and interventionist approaches and calls for an integrated approach to regional security policy. The findings indicate that a synergistic combination of preventive measures focused on radicalization prevention and social cohesion, coupled with sound security strategies, is essential to effectively combat fanatical insurgency, terrorism, brigandage, and narcoterrorism. Therefore, a balanced and integrated approach between preventive and repressive measures is necessary to guarantee security and promote stability in the region.

Research prospects could include an in-depth study of the organization and effectiveness of defense systems in the Sahel, as well as an analysis of administrative and security interlocking patterns to determine the most effective configurations for ensuring security and stability. In addition, the study of conflict dynamics in the region could be undertaken to understand the underlying causes of tension and violence, identifying key actors and their motivations. At the same time, research could be conducted on peace-building strategies to prevent conflict and promote reconciliation, while border and mobility management could be studied to better understand the challenges associated with the movement of people, goods, and weapons across the porous borders of the Sahel.

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