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Stabilization of a population dynamic model with delay term by a boundary dynamical control

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Abstract : We consider a population dynamic system with delay term and nonlocal boundary condition. The semigroup theory combined with the Banach fixed point theorem lead to the well posedness of the problem. Using a boundary control, we establish first the strong stability. After, we prove the uniform stability of the model by using the frequency domain approach.

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1 Introduction

In a natural ecosystem, the investigations on the dynamic properties of the population dynamics may neglect the possible effects induced by time delay. The important effects of time delays in dynamical systems have been brought to light in [11, 24]. Time delay is also very important for issues that affect the survival of humans around the world. For example, the time delay is a key factor in studying the dynamic mechanism of COVID-19 transmission [14, 19]. The authors of [22] considered and analyzed a novel stage-structured single population model with state-dependent maturity delay. In [12], S. Ma and al. investigated a logistic population model with a maturation delay stage for adults. Moreover, Magpantay and Kosovalić considered in [13], an age-structured population model with distinct immature and adult stages, wherein the populations at each stage consume different limited food sources. Properties of solutions to this model are derived and the dynamics are compared to the corresponding constant delay case when state-dependence is ignored. A new age-structured model for a closed population with space-limited recruitment is proposed by

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Angulo and al in [1]. This problem incorporates a time delay in the settlement process representing, for a marine population of invertebrates, the pelagic larval phase previous to the sessile stage. Recently, a novel method for basic reproduction ratio of a size-structured population model with delay by Kumar and al in [10].

In the literature, there are full practical processes that might be modelled by distributed delay systems,which present a wide range of applications in various fields. In the recent past (last four decades), many researchers have developed mathematical tools in order to establish polynomial or exponential decays of these systems. For a list of early works, see [18] and for some other relevant results, we refer readers to [9, 20] and the references therein.

In this paper, we consider the following population dynamics model with delay term

$$\begin{cases} y_t(t,a,x) + y_a(t,a,x) - \Delta y(t,a,x) + \mu(a)y(t,a,x) = 0 & \text{in} \quad (0,+\infty)\times(0,A)\times\Omega, \\ y(t,a,\sigma) = 0 & \text{on} \quad (0,+\infty)\times(0,A)\times\Gamma_D, \\ y_\nu(t,a,\sigma) = u(t,a,\sigma) & \text{on} \quad (0,+\infty)\times(0,A)\times\Gamma_N, \\ y(t,a,x) = y_0(t,a,x) & \text{on} \quad (-\tau,0)\times(0,A)\times\Omega, \\ y(t,0,x) = \displaystyle\int_0^A \beta(a)y(t-\tau,a,x)da & \text{on} \quad (0,+\infty)\times\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary Γ such that $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$; $\tau > 0$ denotes the time delay. Here $y(t,a,x)$ is the distribution of individuals of age a at time t and location $x \in \Omega$. The constant A is the maximum life expectancy, Δ the Laplacian with respect to the spatial variable and $\sigma \in \Gamma$; and $u(t,a,\sigma)$ the control function. The natural fertility and the natural death rate of individuals are respectively denoted by β and μ. We assume that the fertility and the natural death rate satisfy the demographic properties:

$$(H_1)\begin{cases} \mu(a) > 0 \text{ a.e } a \in (0,A), \\ \displaystyle\int_0^A \mu(s)ds = +\infty, \end{cases}$$

and

$$(H_2)\begin{cases} \beta(a) \geq 0 \text{ for every } a \in (0,A), \\ \beta \in L^2(0,A). \end{cases}$$

Let us introduce the so-called *net reproduction rate* $R_0 := \int_0^A \beta(a)exp\big(-\int_0^a \mu(s)ds\big)da$. It is well known that for $\tau = 0$ and $u = 0$ (see for intance [2]), if $R < 1$ then

$$\lim_{T\to+\infty} \|y(T,.,.)\|_{L^2((0,A)\times\Omega)} = 0$$

while if $R > 1$ then

$$\lim_{T\to+\infty} \|y(T,.,.)\|_{L^2((0,A)\times\Omega)} = +\infty.$$

In what follows, we consider a feadback control. More precisely, we take $u = -\eta y,\ \eta > 0$ and we prove that one gets various stablity result even if $R > 1$.
In this recent work, D. Yan [23] investigated the long time behavior for a size-dependent population system with diffusion and Riker type birth function. Some dynamical properties of the considered system is obtained by using C_0-semigroup theory and spectral analysis arguments. Some sufficient conditions are obtained respectively for asymptotical stability, asynchronous exponential growth at the null equilibrium as well as Hopf bifurcation occurring at the positive steady state of the model. In our work, this method is not used.

The main novelties brought in by our paper are enumerated below.

• The first novelty in this work is the introduction of delay term in the model structured by time, age and space with a boundary control on a part of the domain.

• Here, we use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [3, 7, 20, 21]. With this technic, we show that the $\mathcal{C}_0-$semigroup of the system (1.1) is strongly stable on the space $L^2\big((0, A) \times \Omega\big)$.

• Exponential stability of the problem is proved using the frequency domain approach introduce in [8].

The next sections are organized as follows: in Section 2, the system (1.1) is written in a semigroup approach and we obtain the well posedness result. The Section 3 deals with the strong stability of the problem. Then, the uniform stability is established in Section 4.

2 Well posedness

This section is devoted to the study of the well posedness of the problem (1.1) using the semigroup theory combined with the Banach fixed point theorem.
Let us denote by μ_0 a positive constant which will be fixed later. We make the following standard change $\hat{y} = e^{-\mu_0 t} y$. Then, it follows that $\hat{y}$ solves the following system:

$$\begin{cases} \hat{y}_t(t, a, x) + \hat{y}_a(t, a, x) - \Delta \hat{y}(t, a, x) + \big(\mu(a) + \mu_0\big)\hat{y}(t, a, x) = 0 & \text{in} \quad (0, +\infty) \times (0, A) \times \Omega, \\ \hat{y}(t, a, \sigma) = 0 & \text{on} \quad (0, +\infty) \times (0, A) \times \Gamma_D, \\ \hat{y}_\nu(t, a, \sigma) + \eta \hat{y}(t, a, \sigma) = 0 & \text{on} \quad (0, +\infty) \times (0, A) \times \Gamma_N, \\ \hat{y}(t, a, x) = \hat{y}_0(t, a, x) & \text{on} \quad (-\tau, 0) \times (0, A) \times \Omega, \\ \hat{y}(t, 0, x) = \displaystyle\int_0^A \beta(a)\hat{y}(t - \tau, a, x)da & \text{on} \quad (0, +\infty) \times \Omega, \end{cases} \tag{2.1}$$

where $\hat{y}_0(t, a, x) = e^{-\mu_0 t} y_0(t, a, x)$ is the initial density of the population.
The existence of solution of problem (1.1) is now reduced to the well posedness of the problem (2.1).
Let us set

$$p(t, a, x, \rho) = \hat{y}(t - \rho\tau, a, x),\ (t, a, x, \rho) \in (0, +\infty) \times (0, A) \times \Omega \times (0, 1). \tag{2.2}$$

Then, we obtain

$$\begin{cases} p_t(t,a,x,\rho)+\frac{1}{\tau}p_\rho(t,a,x,\rho)=0 & \text{in} \quad (0,+\infty)\times(0,A)\times\Omega\times(0,1), \\ p(0,a,x,\rho)=\hat{y}_0(-\rho\tau,a,x) & \text{on} \quad (0,A)\times\Omega\times(0,1), \\ p(t,a,x,0)=\hat{y}(t,a,x) & \text{on} \quad (0,+\infty)\times(0,A)\times\Omega, \\ p(t,a,x,1)=\hat{y}(t-\tau,a,x) & \text{on} \quad (0,+\infty)\times(0,A)\times\Omega. \end{cases}$$

The problem (2.1) is now equivalent to the following

$$\begin{cases} \hat{y}_t(t,a,x)+\hat{y}_a(t,a,x)-\Delta\hat{y}(t,a,x)+\big(\mu(a)+\mu_0\big)\hat{y}(t,a,x)=0 & \text{in} \quad (0,+\infty)\times(0,A)\times\Omega, \\ \hat{y}(t,a,\sigma)=0 & \text{on} \quad (0,+\infty)\times(0,A)\times\Gamma_D, \\ \hat{y}_\nu(t,a,\sigma)+\eta\hat{y}(t,a,\sigma)=0 & \text{on} \quad (0,+\infty)\times(0,A)\times\Gamma_N, \\ \hat{y}(t,a,x)=\hat{y}_0(t,a,x) & \text{on} \quad (-\tau,0)\times(0,A)\times\Omega, \\ \hat{y}(t,0,x)=\displaystyle\int_0^A\beta(a)p(t,a,x,1)da & \text{on} \quad (0,+\infty)\times\Omega, \end{cases} \tag{2.3}$$

and

$$\begin{cases} p_t(t,a,x,\rho)+\frac{1}{\tau}p_\rho(t,a,x,\rho)=0 & \text{in} \quad (0,+\infty)\times(0,A)\times\Omega\times(0,1), \\ p(0,a,x,\rho)=\hat{y}_0(-\rho\tau,a,x) & \text{on} \quad (0,A)\times\Omega\times(0,1), \\ p(t,a,x,0)=\hat{y}(t,a,x) & \text{on} \quad (0,+\infty)\times(0,A)\times\Omega, \\ p(t,a,x,1)=\hat{y}(t-\tau,a,x) & \text{on} \quad (0,+\infty)\times(0,A)\times\Omega. \end{cases} \tag{2.4}$$

If we denote by

$$X=\begin{pmatrix} \hat{y} \\ p \end{pmatrix},$$

one has from (2.3) and (2.4)

$$X_t=\begin{pmatrix} \hat{y}_t \\ p_t \end{pmatrix}=\begin{pmatrix} -\hat{y}_a+\Delta\hat{y}-\big(\mu+\mu_0\big)\hat{y} \\ \frac{-1}{\tau}p_\rho \end{pmatrix}=\mathcal{A}X,$$

with the domain

$$D(\mathcal{A})=\Bigg\{\begin{pmatrix} \hat{y} \\ p \end{pmatrix}\Bigg|\ \hat{y}\in L^2\Big((0,A);H^2(\Omega)\cap\mathcal{V}\Big)\cap H^1\Big((0,A);L^2(\Omega)\Big);\ p\in H^1\Big((0,1);L^2(\Omega)\Big);$$
$$\hat{y}(0,x)=\int_0^A\beta(a)p(a,x,1)da;\ p(a,x,0)=\hat{y}(a,x)\Bigg\}$$

and

$$\mathcal{V}=\left\{\hat{y}\in L^2\Big((0,A);H^1(\Omega)\Big)\Big|\ \hat{y}(a,\sigma)=0 \text{ on } (0,A)\times\Gamma_D \text{ and } \hat{y}_\nu(a,\sigma)=-\eta\hat{y}(a,\sigma) \text{ on } (0,A)\times\Gamma_N\right\}.$$

Denote by $\mathcal{H}$ the Hilbert space as below

$$\mathcal{H}=L^2\big((0,A)\times\Omega\big)\times L^2\big((0,A)\times\Omega\times(0,1)\big)$$

endowed with the inner product

$$\left\langle\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\Big|\begin{pmatrix}\hat{z}\\ q\end{pmatrix}\right\rangle=\int_0^A\int_\Omega \hat{y}\bar{\hat{z}}dxda+\gamma\int_0^A\int_\Omega\int_0^1 p\bar{q}d\rho dxda$$

where γ is a positive constant.
Now, we can state the existence result.

Theorem 2.1. *Under the assumptions* (H_1) *and* (H_2), *for all* $X_0=(\hat{y}_0,p_0)\in D(\mathcal{A})$, *the problem* $(2.3)-(2.4)$ *has a unique solution* $(\hat{y},p)$ *which satisfies:*

$$(\hat{y},p)\in C\big([0;\infty)\times[0;\infty);D(\mathcal{A})\big)\cap C^1\big([0;\infty)\times[0;\infty);\mathcal{H}\big).$$

Proof. (Theorem 2.1)
Step 1: $\mathcal{A}$ **is dissipative**.
Let us consider $(\hat{y},p)^T\in D(\mathcal{A})$

$$\left\langle\mathcal{A}\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\Big|\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\right\rangle=\int_0^A\int_\Omega\big[-\hat{y}_a+\Delta\hat{y}-(\mu+\mu_0)\hat{y}\big]\bar{\hat{y}}dxda-\frac{\gamma}{\tau}\int_0^A\int_\Omega\int_0^1 p_\rho\bar{p}d\rho dxda$$

So,

$$\begin{aligned}\Re e\left(\left\langle\mathcal{A}\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\Big|\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\right\rangle\right)=&\frac{1}{2}\int_\Omega|\hat{y}(0,x)|^2dx-\eta\int_0^A\int_{\Gamma_N}|\hat{y}(a,\sigma)|^2d\sigma da-\int_0^A\int_\Omega\big|\nabla\hat{y}(a,x)\big|^2dxda\\&-\int_0^A\int_\Omega(\mu+\mu_0)|\hat{y}(a,x)|^2dxda-\frac{\gamma}{2\tau}\int_0^A\int_\Omega|p(a,x,1)|^2dxda\\&+\frac{\gamma}{2\tau}\int_0^A\int_\Omega|p(a,x,0)|^2dxda\\=&\frac{1}{2}\int_\Omega\left|\int_0^A\beta(a)p(a,x,1)da\right|^2dx-\eta\int_0^A\int_{\Gamma_N}|\hat{y}(a,\sigma)|^2d\sigma da-\int_0^A\int_\Omega\big|\nabla\hat{y}(a,x)\big|^2dxda\\&-\int_0^A\int_\Omega(\mu+\mu_0)|\hat{y}(a,x)|^2dxda-\frac{\gamma}{2\tau}\int_0^A\int_\Omega|p(a,x,1)|^2dxda\\&+\frac{\gamma}{2\tau}\int_0^A\int_\Omega|\hat{y}(a,x)|^2dxda.\end{aligned}$$

By using Cauchy-Schwarz inegality, it follows that

$$\frac{1}{2}\int_\Omega\left|\int_0^A\beta(a)p(a,x,1)da\right|^2dx-\frac{\gamma}{2\tau}\int_0^A\int_\Omega|p(a,x,1)|^2dxda\leq\left(\frac{1}{2}\int_0^A|\beta(a)|^2da-\frac{\gamma}{2\tau}\right)\int_0^A\int_\Omega|p(a,x,1)|^2dxda.$$

Then,

$$
\begin{aligned}
\Re e\left(\left\langle \mathcal{A}\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\middle|\begin{pmatrix}\hat{y}\\ p\end{pmatrix}\right\rangle\right) \leq & -\eta\int_0^A\int_{\Gamma_N}|\hat{y}(a,\sigma)|^2 d\sigma da - \int_0^A\int_\Omega \left|\nabla\hat{y}(a,x)\right|^2 dxda \\
& -\left(\mu_0-\frac{\gamma}{2\tau}\right)\int_0^A\int_\Omega|\hat{y}(a,x)|^2 dxda \\
& +\left(\frac{1}{2}\int_0^A|\beta(a)|^2 da-\frac{\gamma}{2\tau}\right)\int_0^A\int_\Omega|p(a,x,1)|^2 dxda.
\end{aligned}
$$

Therefore, choosing $\mu_0 = \dfrac{\gamma}{2\tau}+1$ and $\dfrac{\gamma}{2\tau} = \dfrac{1}{2}\displaystyle\int_0^A|\beta(a)|^2 da+1$, we get that $\mathcal{A}$ is dissipative.

Step 2: $\lambda\mathbf{I}-\mathcal{A}$ **is surjective for at least one** $\lambda > \mathbf{0}$.

Let $(\hat{z},q)\in\mathcal{H}$. The equality $(\lambda I-\mathcal{A})(\hat{y},p)^T = (\hat{z},q)^T$ implies

$$
\begin{cases}
\hat{y}_a(a,x)-\Delta\hat{y}(a,x)+\big(\lambda+\mu(a,x)+\mu_0\big)\hat{y}(a,x)=\hat{z}(a,x) & \text{in} \quad (0,A)\times\Omega,\\
\hat{y}(a,\sigma)=0 & \text{on} \quad (0,A)\times\Gamma_D,\\
\hat{y}_\nu(a,\sigma)=-\eta\hat{y}(a,\sigma) & \text{on} \quad (0,A)\times\Gamma_N,\\
\hat{y}(0,x)=\displaystyle\int_0^A\beta(a)p(a,x,1)da & \text{on} \quad \Omega,
\end{cases}
\tag{2.5}
$$

and

$$
\begin{cases}
p_\rho(a,x,\rho)+\lambda\tau p(a,x,\rho)=\tau q(a,x,\rho) & \text{in} \quad (0,A)\times\Omega\times(0,1),\\
p(a,x,0)=\hat{y}(a,x) & \text{on} \quad (0,A)\times\Omega.
\end{cases}
\tag{2.6}
$$

Now, we consider the following auxiliary problem obtaining from (2.5)

$$
\begin{cases}
\hat{y}_a(a,x)-\Delta\hat{y}(a,x)+\big(\lambda+\mu(a,x)+\mu_0\big)\hat{y}(a,x)=\hat{z}(a,x) & \text{in} \quad (0,A)\times\Omega,\\
\hat{y}(a,\sigma)=0 & \text{on} \quad (0,A)\times\Gamma_D,\\
\hat{y}_\nu(a,\sigma)=-\eta\hat{y}(a,\sigma) & \text{on} \quad (0,A)\times\Gamma_N,\\
\hat{y}(0,x)=\displaystyle\int_0^A\beta(a)\Theta(a,x)da & \text{on} \quad \Omega.
\end{cases}
\tag{2.7}
$$

It is obvious that the problem (2.7) admits a unique solution. Consequently, the solution of (2.6) is given by

$$
p(a,x,\rho)=\hat{y}(a,x)e^{-\lambda\tau\rho}+\hat{y}(a,x)e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau s}\tau q(a,x,s)ds. \tag{2.8}
$$

Let us define $\Phi\ :\ \Theta\mapsto y\mapsto p(a,x,1)$. The goal is to prove that Φ is a contraction. Setting

$\hat{Y} = \hat{y}_1 - \hat{y}_2,\ P = p_1 - p_2,\ \theta = \Theta_1 - \Theta_2$; then $\hat{Y}$ and P are solution of

$$\begin{cases} \hat{Y}_a(a,x) - \Delta\hat{Y}(a,x) + \big(\lambda + \mu(a,x) + \mu_0\big)\hat{Y}(a,x) = 0 & \text{in} \quad (0,A)\times\Omega, \\ \hat{Y}(a,\sigma) = 0 & \text{on} \quad (0,A)\times\Gamma_D, \\ \hat{Y}_\nu(a,\sigma) = -\eta\hat{Y}(a,\sigma) & \text{on} \quad (0,A)\times\Gamma_N, \\ \hat{Y}(0,x) = \displaystyle\int_0^A \beta(a)\theta(a,x)da & \text{on} \quad \Omega, \end{cases} \tag{2.9}$$

and

$$\begin{cases} P_\rho(a,x,\rho) + \lambda\tau P(a,x,\rho) = 0 & \text{in} \quad (0,A)\times\Omega\times(0,1), \\ P(a,x,0) = \hat{Y}(a,x) & \text{on} \quad (0,A)\times\Omega. \end{cases} \tag{2.10}$$

Multiplying the first equation of (2.9) by $\bar{\hat{Y}}$ and integrating by parts on $(0,A)\times\Omega$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\hat{Y}(A,x)|^2dx + \eta\int_0^A\int_{\Gamma_N}|\hat{Y}(a,\sigma)|^2d\sigma da + \int_0^A\int_\Omega \left|\nabla\hat{Y}(a,x)\right|^2dxda \\ &+ \int_0^A\int_\Omega(\lambda+\mu+\mu_0)|\hat{Y}(a,x)|^2dxda = \frac{1}{2}\int_\Omega\left|\int_0^A\beta(a)\theta(a,x)da\right|^2dx. \end{aligned} \tag{2.11}$$

Using (2.11) and the Cauchy-Schwarz inequality, we get

$$\int_0^A\int_\Omega(\lambda+\mu_0)|\hat{Y}(a,x)|^2dxda \leq \frac{1}{2}\int_0^A|\beta(a)|^2da\int_0^A\int_\Omega|\theta(a,x)|^2dxda.$$

Let us choose $\lambda + \mu_0$ such that $\lambda + \mu_0 > 1$, then

$$\int_0^A\int_\Omega|\hat{Y}(a,x)|^2dxda \leq \frac{1}{2}\int_0^A|\beta(a)|^2da\int_0^A\int_\Omega|\theta(a,x)|^2dxda.$$

From (2.8), we have $P(a,x,1) = \hat{Y}(a,x)e^{-\lambda\tau}$. So

$$\int_0^A\int_\Omega|P(a,x,1)|^2dxda \leq \frac{C}{e^{2\lambda\tau}}\int_0^A\int_\Omega|\theta(a,x)|^2dxda$$

where $C = \dfrac{1}{2}\displaystyle\int_0^A|\beta(a)|^2da$. That is

$$\left|\left|p_1 - p_2\right|\right|_{L^2((0,A)\times\Omega)} \leq \frac{\sqrt{C}}{e^{\lambda\tau}}\left|\left|\Theta_1 - \Theta_2\right|\right|_{L^2((0,A)\times\Omega)}.$$

Choosing λ large enough, it follows that Φ is a contraction. Thus the problem (2.5) – (2.6) admits a unique solution by the Banach fixed point theorem. Consequently $\lambda I - \mathcal{A}$ is surjective.
Since $\overline{D(\mathcal{A})} = \mathcal{H}$ (see [5]), then using Lumer-Phillips theorem (see [16]) the operator $\mathcal{A}$ generates a $\mathcal{C}_0$ semigroup of contraction in $\mathcal{H}$. Consequently, we get the existence of solution of problem (2.1) see for instance [6, 16].

3 Strong stability

The main result of this section is given as follows.

Theorem 3.1. *For all $y_0 \in L^2\big((0,A)\times\Omega\big)$, The $\mathcal{C}_0-$semigroup of system* (1.1) *is strongly stable on the space $L^2\big((0,A)\times\Omega\big)$.*

We need the following result for the proof of Theorem 3.1.

Theorem 3.2. *The $\mathcal{C}_0-$semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the space $\mathcal{H}$. That is:*

$$\forall U_0 = (\hat{y}_0, p_0) \in \mathcal{H}, \ \lim_{t\to+\infty} ||e^{t\mathcal{A}}U_0||_{\mathcal{H}} = 0.$$

For the proof of Theorem 3.2, we state the following results.

Lemma 3.3. *The resolvant of the operator $\mathcal{A}$ is compact.*

Proof. (Lemma 3.3)
Let λ be on the resolvant set of $\mathcal{A}$, $(f,g)\in\mathcal{H}$ and $(\hat{y},p)\in D(\mathcal{A})$. Then $(\lambda I-\mathcal{A})(\hat{y},p)^T = (f,g)^T$ may be written as:

$$\begin{cases} \hat{y}_a(a,x) - \Delta\hat{y}(a,x) + \big(\lambda+\mu(a,x)+\mu_0\big)\hat{y}(a,x) = f(a,x) & \text{in} \quad (0,A)\times\Omega, \\ \hat{y}(a,\sigma) = 0 & \text{on} \quad (0,A)\times\Gamma_D, \\ \hat{y}_\nu(a,\sigma) = -\eta\hat{y}(a,\sigma) & \text{on} \quad (0,A)\times\Gamma_N, \\ \hat{y}(0,x) = \displaystyle\int_0^A \beta(a)p(a,x,1)da & \text{on} \quad \Omega, \end{cases} \tag{3.1}$$

and

$$\begin{cases} p_\rho(a,x,\rho) + \lambda\tau p(a,x,\rho) = \tau g(a,x,\rho) & \text{in} \quad (0,A)\times\Omega\times(0,1), \\ p(a,x,0) = \hat{y}(a,x) & \text{on} \quad (0,A)\times\Omega. \end{cases} \tag{3.2}$$

Multiplying the first equation of (3.1) by $\bar{\hat{y}}$ and integrating by parts on $(0,A)\times\Omega$, we obtain

$$\int_0^A\int_\Omega(\lambda+\mu+\mu_0)|\hat{y}|^2dxda + \frac{1}{2}\int_0^A\int_\Omega\big|\nabla\hat{y}\big|^2dxda + \eta\int_0^A\int_{\Gamma_N}|\hat{y}|^2d\sigma da = \frac{1}{2}\int_\Omega\left|\int_0^A\beta(a)p(a,x,1)da\right|^2dx \tag{3.3}$$
$$+\int_0^A\int_\Omega f\bar{\hat{y}}dxda.$$

Using (3.3), Cauchy-Schwarz and young inequalities, it follows that

$$\lambda\int_0^A\int_\Omega|\hat{y}|^2dxda \leq \frac{1}{2}\int_0^A\int_\Omega|f|^2dxda + \frac{1}{2}\int_0^A\int_\Omega|\hat{y}|^2dxda + \frac{1}{2}\int_0^A|\beta(a)|^2da\int_0^A\int_\Omega|p(a,x,1)|^2dxda.$$

Thus

$$\left(\lambda-\frac{1}{2}\right)\int_0^A\int_\Omega|\hat{y}|^2dxda\leq\frac{1}{2}\int_0^A\int_\Omega|f|^2dxda+\frac{1}{2}\int_0^A|\beta(a)|^2da\int_0^A\int_\Omega|p(a,x,1)|^2dxda. \tag{3.4}$$

Moreover, from (3.2) we get

$$p(a,x,\rho)=\hat{y}(a,x)e^{-\lambda\tau\rho}+\hat{y}(a,x)e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau s}\tau g(a,x,s)ds. \tag{3.5}$$

The equality (3.5) implies that

$$p(a,x,1)=\hat{y}(a,x)e^{-\lambda\tau}+\hat{y}(a,x)e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds.$$

From the last equality, we obtain

$$\begin{aligned}\int_0^A\int_\Omega|p(a,x,1)|^2dxda&=\int_0^A\int_\Omega|\hat{y}|^2e^{-2\lambda\tau}dxda+\int_0^A\int_\Omega\left|\hat{y}e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds\right|^2dxda \\ &+2\int_0^A\int_\Omega\left(\hat{y}e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds\right)dxda.\end{aligned} \tag{3.6}$$

Furthermore, by using Cauchy-Schwarz and Young inequalities, we have

$$\int_0^A\int_\Omega\left|\hat{y}e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds\right|^2dxda\leq\frac{1}{2}\int_0^A\int_\Omega|\hat{y}|^2e^{-2\lambda\tau}dxda+\frac{\tau^2}{2}\int_0^A\int_\Omega\int_0^1 e^{2\lambda\tau s}|g|^2dsdxda \tag{3.7}$$

and

$$2\int_0^A\int_\Omega\left(\hat{y}e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds\right)dxda\leq\int_0^A\int_\Omega\left[|\hat{y}|^2e^{-2\lambda\tau}+\left|\hat{y}e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds\right|^2\right]dxda.$$

The last inequality and (3.7) leads to

$$2\int_0^A\int_\Omega\left(\hat{y}e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}\tau g(a,x,s)ds\right)dxda\leq\frac{3}{2}\int_0^A\int_\Omega|\hat{y}|^2e^{-2\lambda\tau}dxda+\frac{\tau^2}{2}\int_0^A\int_\Omega\int_0^1 e^{2\lambda\tau s}|g|^2dsdxda. \tag{3.8}$$

Finally, (3.4), (3.6), (3.7) and (3.8) give

$$\left(2\lambda-1-3\int_0^A|\beta(a)|^2da\right)\int_0^A\int_\Omega|\hat{y}|^2dxda\leq\int_0^A\int_\Omega|f|^2dxda+\int_0^A\tau^2|\beta(a)|^2da\int_0^A\int_\Omega\int_0^1 e^{2\lambda\tau s}|g|^2dsdxda.$$

Choosing λ large enough, it follows that

$$\int_0^A\int_\Omega|\hat{y}|^2dxda\leq\int_0^A\int_\Omega|f|^2dxda+\int_0^A\tau^2\beta^2da\int_0^A\int_\Omega\int_0^1 e^{2\lambda\tau s}|g|^2dsdxda. \tag{3.9}$$

In the same way, we obtain

$$\int_0^A \int_\Omega \int_0^1 |p|^2 d\rho dx da \leq 3 \int_0^A \int_\Omega |\hat{y}|^2 dx da + \int_0^A \int_\Omega \int_0^1 e^{2\lambda\tau s} \tau^2 |g|^2 d\rho dx da.$$

So

$$\int_0^A \int_\Omega \int_0^1 |p|^2 d\rho dx da \leq 3 \int_0^A \int_\Omega |f|^2 dx da + \left(3 \int_0^A \tau^2 |\beta|^2 da + \tau^2\right) \int_0^A \int_\Omega \int_0^1 e^{2\lambda\tau s} |g|^2 d\rho dx da. \tag{3.10}$$

Now, for $((f_n, g_n))_n$ a bounded sequence in $\mathcal{H}$ we see from (3.9) and (3.10) that the corresponding solution $((\hat{y}_n, p_n))_n$ is bounded in $D(\mathcal{A})$. Hence $((\hat{y}_n, p_n))_n$ has a convergence subsequence in $\mathcal{H}$. Thus $(\lambda I - \mathcal{A})^{-1}$ is a compact operator.

Lemma 3.4. *There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. (Lemma 3.4)
By contradiction argument, we assume that there exists at least one $i\lambda \in \sigma(\mathcal{A})$, $\lambda \in \mathbb{R}^*$ on the imaginary axis. Let $V = (\hat{y}, p)^T \in D(\mathcal{A})$ be the corresponding eigenvector such that $||V|| = 1$ and

$$\mathcal{A}V = i\lambda V,$$

which is equivalent to

$$\begin{cases} -\hat{y}_a + \Delta\hat{y} - (i\lambda + \mu + \mu_0)\hat{y} = 0 & \text{in } (0, A) \times \Omega, \\ p_\rho + i\lambda\tau p = 0 & \text{in } (0, A) \times \Omega \times (0, 1), \\ \hat{y}(a, \sigma) = 0 & \text{on } (0, A) \times \Gamma_D, \\ \hat{y}_\nu(a, \sigma) = -\eta\hat{y}(a, \sigma) & \text{on } (0, A) \times \Gamma_N, \\ \hat{y}(0, x) = \displaystyle\int_0^A \beta(a) p(a, x, 1) da & \text{on } \Omega, \\ p(a, x, 0) = \hat{y}(a, x) & \text{on } (0, A) \times \Omega. \end{cases} \tag{3.11}$$

Recalling the dissipativity of $\mathcal{A}$ in the proof of Theorem 2.1, it follows that

$$0 = \Re e\langle \mathcal{A}V | V \rangle \leq -\eta \int_0^A \int_{\Gamma_N} |\hat{y}(a, \sigma)|^2 d\sigma da - \int_0^A \int_\Omega \left|\nabla\hat{y}(a, x)\right|^2 dx da - \left(\mu_0 - \frac{\gamma}{2\tau}\right) \int_0^A \int_\Omega |\hat{y}(a, x)|^2 dx da \tag{3.12}$$

$$+ \left(\frac{1}{2} \int_0^A |\beta(a)|^2 da - \frac{\gamma}{2\tau}\right) \int_0^A \int_\Omega |p(a, x, 1)|^2 dx da \leq 0.$$

That is $\hat{y}(a, x) = 0$ for almost every(a.e.) $(a, x) \in (0, A) \times \Omega$. and using the second and the last equalities of (3.11) we get $p(a, x, \rho) = 0$ *a.e.* $(a, x, \rho) \in (0, A) \times \Omega \times (0, 1)$. Which contradicts the fact that $||V|| = 1$. We conclude that $\mathcal{A}$ has no eigenvalue on the imaginary axis.

Proof. (Theorem 3.2)

We use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [3, 7, 21]. The resolvant of $\mathcal{A}$ is compact from Lemma 3.3 and $\mathcal{A}$ has no eigenvalue on the imaginary axis from Lemma 3.4. Following this theory, the conditions of the spectral decomposition theory of Sz-Nagy-Foias and Foguel are satisfied. So, we get the desired result.

Proof. (Theorem 3.1)

Let us define the operator $\mathcal{A}_0$ on $\mathcal{H}$ by:

$$\mathcal{A}_0\begin{pmatrix}\hat{y}\\ p\end{pmatrix}=\begin{pmatrix}-\hat{y}_a+\Delta\hat{y}-\mu\hat{y}\\ \frac{-1}{\tau}p_\rho\end{pmatrix}=\mathcal{A}\begin{pmatrix}\hat{y}\\ p\end{pmatrix}+\tilde{\mathcal{A}}\begin{pmatrix}\hat{y}\\ p\end{pmatrix},$$

with $\tilde{\mathcal{A}}\begin{pmatrix}\hat{y}\\ p\end{pmatrix}=\begin{pmatrix}\mu_0\hat{y}\\ 0\end{pmatrix}$.

Step 1: $\mathcal{A}_0$ generates a $\mathcal{C}_0-$ semigroup and has a compact resolvent.

From the section 2, the operator $\mathcal{A}$ is an infinitesimal generator of a $\mathcal{C}_0-$semigroup. Moreover, $\tilde{\mathcal{A}}$ is a bounded operator. Thus, from Lemma 3.2 of [3], $\mathcal{A}_0$ generates a $\mathcal{C}_0-$semigroup and has a compact resolvent.

Step 2: $\mathcal{A}_0$ has no eigenvalue on the imaginary axis.

By contradiction argument, we assume that there exists at least one $i\lambda\in\sigma(\mathcal{A}_0),\ \lambda\in\mathbb{R}^*$ on the imaginary axis. Let $V=(\hat{y},p)^T\in D(\mathcal{A}_0)$ be the corresponding eigenvector such that $||V||=1$ and

$$\mathcal{A}_0V=i\lambda V,$$

which is equivalent to

$$\begin{cases}-\hat{y}_a+\Delta\hat{y}-(i\lambda+\mu)\hat{y}=0 & \text{in } (0,A)\times\Omega,\\ p_\rho+i\lambda\tau p=0 & \text{in } (0,A)\times\Omega\times(0,1),\\ \hat{y}(a,\sigma)=0 & \text{on } (0,A)\times\Gamma_D,\\ \hat{y}_\nu(a,\sigma)=-\eta\hat{y}(a,\sigma) & \text{on } (0,A)\times\Gamma_N,\\ \hat{y}(0,x)=\displaystyle\int_0^A\beta(a)p(a,x,1)da & \text{on } \Omega,\\ p(a,x,0)=\hat{y}(a,x) & \text{on } (0,A)\times\Omega.\end{cases}\tag{3.13}$$

We have

$$0=\Re e\big(\langle\mathcal{A}_0V|V\rangle\big)=\Re e\big(\langle\mathcal{A}V|V\rangle\big)+\Re e\big(\langle\tilde{\mathcal{A}}V|V\rangle\big)=\Re e\big(\langle\mathcal{A}V|V\rangle\big).\tag{3.14}$$

Using (3.14) and recalling the dissipativity of $\mathcal{A}$ in the proof of Theorem 2.1, it follows that

$$\begin{aligned} 0 = \Re e\big(\langle \mathcal{A}_0 V|V\rangle\big) = \Re e\big(\langle \mathcal{A}V|V\rangle\big) \leq & -\eta \int_0^A \int_{\Gamma_N} |\hat{y}(a,\sigma)|^2 d\sigma da - \int_0^A \int_\Omega \big|\nabla \hat{y}(a,x)\big|^2 dxda \\ & - \left(\mu_0 - \frac{\gamma}{2\tau}\right) \int_0^A \int_\Omega |\hat{y}(a,x)|^2 dxda \\ & + \left(\frac{1}{2}\int_0^A |\beta(a)|^2 da - \frac{\gamma}{2\tau}\right) \int_0^A \int_\Omega |p(a,x,1)|^2 dxda \leq 0. \end{aligned}$$

That is $\hat{y}(a,x) = 0$ $a.e.$ $(a,x) \in (0,A)\times\Omega$. and using the second and the last equalities of (3.13) we get $p(a,x,\rho) = 0$ $a.e.$ $(a,x,\rho) \in (0,A)\times\Omega\times(0,1)$. Which contradicts the fact that $||V|| = 1$. We conclude that $\mathcal{A}_0$ has no eigenvalue on the imaginary axis.
Using the spectral decomposition theory of Sz-Nagy-Foias and Foguel, we conclude that the C_0-semigroup $(e^{t\mathcal{A}_0})_{t\geq 0}$ is strongly stable on the space $\mathcal{H}$. That is for all $U_0 = (\hat{y}_0, p_0) \in \mathcal{H}$, $\lim\limits_{t\to+\infty} ||e^{t\mathcal{A}_0}U_0||_{\mathcal{H}} = 0$.
Furthermore, one can remark that

$$\mathcal{A}_0 \begin{pmatrix} \hat{y} \\ p \end{pmatrix} = \begin{pmatrix} -\hat{y}_a + \Delta\hat{y} - \mu\hat{y} \\ \frac{-1}{\tau}p_\rho \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{A}}_{\prime}^{\infty} & 0 \\ 0 & \tilde{\mathcal{A}}_{\prime}^{\in} \end{pmatrix} \begin{pmatrix} \hat{y} \\ p \end{pmatrix}.$$

Here, $\tilde{\mathcal{A}}_{\prime}^{\infty}\hat{y} = -\hat{y}_a + \Delta\hat{y} - \mu\hat{y}$ and $\tilde{\mathcal{A}}_{\prime}^{\in} p = \frac{-1}{\tau}p_\rho$. Thus,

$$\lim_{t\to+\infty} ||e^{t\mathcal{A}_0}U_0||_{\mathcal{H}} = 0 \Rightarrow \lim_{t\to+\infty} ||e^{t\tilde{\mathcal{A}}_{\prime}^{\infty}} y_0||_{\mathcal{H}} = 0.$$

So, the $\mathcal{C}_0-$semigroup of system (1.1) is strongly stable on the space $L^2\big((0,A)\times\Omega\big)$.

4 Exponential stability

Here, the goal is to show that the semigroup generated by the operator of system (1.1) is exponentially stable. For that we use the frequency domain approach, namely the below result.

Lemma 4.1. *[8, 17] A $\mathcal{C}_0-$semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of contraction on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable that is,*

$$||e^{t\mathcal{A}}U_0||_{\mathcal{H}} \leq Ce^{-\omega t}||U_0||_{\mathcal{H}}, \quad \forall\, U_0 \in \mathcal{H},\ \forall\, t \geq 0, \tag{4.1}$$

for some positive constants C and ω, if and only if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{4.2}$$

and

$$\sup_{\beta\in\mathbb{R}} \big|\big|(i\beta I - \mathcal{A})^{-1}\big|\big|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.3}$$

$\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

We have the following result.

Theorem 4.2. *The problem* $(2.3)-(2.4)$ *is exponentially stable in the space* $\mathcal{H}$.

Proof. (Theorem 4.2)
From Lemma 3.4 the condition (4.2) is satisfied. Now, we will prove the condition (4.3).
Let $\beta \in \mathbb{R}$ and $F=(f,g)\in\mathcal{H}$. The solution $U=(\hat{y},p)\in D(\mathcal{A})$ of the system $(i\beta I-\mathcal{A})U^T=F^T$ can be written by

$$\begin{cases} \hat{y}_a(a,x)-\Delta\hat{y}(a,x)+\big(i\beta+\mu(a)+\mu_0\big)\hat{y}(a,x)=f(a,x) & \text{in} \quad (0,A)\times\Omega, \\ \hat{y}(a,\sigma)=0 & \text{on} \quad (0,A)\times\Gamma_D, \\ \hat{y}_\nu(a,\sigma)=-\eta\hat{y}(a,\sigma) & \text{on} \quad (0,A)\times\Gamma_N, \\ \hat{y}(0,x)=\displaystyle\int_0^A \beta(a)p(a,x,1)da & \text{on} \quad \Omega, \end{cases} \tag{4.4}$$

and

$$\begin{cases} p_\rho(a,x,\rho)+i\beta\tau p(a,x,\rho)=\tau g(a,x,\rho) & \text{in} \quad (0,A)\times\Omega\times(0,1), \\ p(a,x,0)=\hat{y}(a,x) & \text{on} \quad (0,A)\times\Omega. \end{cases} \tag{4.5}$$

Thus, we will prove that $||U||_{\mathcal{H}}\le C||F||_{\mathcal{H}}$; where C a positive constant.
Remark that

$$\Re e\Big(\int_0^A\int_\Omega(i\beta+\mu+\mu_0)|\hat{y}|^2dxda\Big)=\int_0^A\int_\Omega(\mu+\mu_0)|\hat{y}|^2dxda.$$

So, using the same calculations as in the proof of Lemma 3.3 and choosing μ_0 large enough, it follows that

$$\int_0^A\int_\Omega|\hat{y}|^2dxda\le\int_0^A\int_\Omega|f|^2dxda+\int_0^A\tau^2|\beta|^2da\int_0^A\int_\Omega\int_0^1|g|^2dsdxda \tag{4.6}$$

and

$$\gamma\int_0^A\int_\Omega\int_0^1|p|^2dsdxda\le 3\gamma\int_0^A\int_\Omega|f|^2dxda+\gamma\Big(3\int_0^A\tau^2|\beta|^2da+\tau^2\Big)\int_0^A\int_\Omega\int_0^1|g|^2dsdxda. \tag{4.7}$$

The inequalities (4.6), (4.7) and $\displaystyle\int_0^A\tau^2|\beta|^2da=\tau\gamma-2\tau^2$ lead to

$$\int_0^A\int_\Omega|\hat{y}|^2dxda\le\int_0^A\int_\Omega|f|^2dxda+\tau\gamma\int_0^A\int_\Omega\int_0^1|g|^2dsdxda \tag{4.8}$$

and

$$\gamma\int_0^A\int_\Omega\int_0^1|p|^2dsdxda\le 3\gamma\int_0^A\int_\Omega|f|^2dxda+\gamma(3\tau\gamma)\int_0^A\int_\Omega\int_0^1|g|^2dsdxda. \tag{4.9}$$

From (4.8), (4.9) and setting $C=2\max\big\{\max\{1,\tau\},\max\{3\gamma,3\gamma\tau\}\big\}$ we obtain the desired result, that is $||U||_{\mathcal{H}}\le C||F||_{\mathcal{H}}$. Therefore $\big|\big|(i\beta I-\mathcal{A})^{-1}\big|\big|$ is bounded. From Lemma 4.1, the semigroup of problem $(2.3)-(2.4)$ is exponentially stable in the space $\mathcal{H}$.

Theorem 4.3. *Assume that*

$$\mu(a) > \frac{1}{2}\Big(1 + 3||\beta||_{L^2(0,A)}\Big), \ \forall\, a \in (0, A). \tag{4.10}$$

Then the problem (1.1) *is exponentially stable in the space* $L^2\big((0, A) \times \Omega\big)$.

Proof. (Theorem 4.3)
Let $\mathcal{B}_0\hat{y} = -\hat{y}_a + \Delta\hat{y} - \mu\hat{y}$. From Theorem 3.1, $i\mathbb{R} \subset \rho(\mathcal{B}_0)$. Consider $\beta \in \mathbb{R}$ and f a function in $L^2\big((0, A) \times \Omega\big)$ such that for $\hat{y} \in D(\mathcal{B}_0)$, $(i\beta I - \mathcal{B}_0)\hat{y} = f$. The objectif is to prove that

$$||\hat{y}||_{L^2\big((0,A)\times\Omega\big)} \le C||f||_{L^2\big((0,A)\times\Omega\big)};$$

where C a positive constant. We have

$$\Re e\bigg(\int_0^A \int_\Omega (i\beta + \mu)|\hat{y}|^2 dxda\bigg) = \int_0^A \int_\Omega \mu|\hat{y}|^2 dxda.$$

Using the same calculations as in the proof of Lemma 3.3, we get

$$\int_0^A \int_\Omega \Big(\mu - \frac{1}{2} - \frac{3}{2}||\beta||_{L^2(0,A)}\Big)|\hat{y}|^2 dxda \le \int_0^A \int_\Omega |f|^2 dxda + \int_0^A \tau^2\beta^2 da \int_0^A \int_\Omega \int_0^1 |g|^2 dsdxda. \tag{4.11}$$

By choosing $g = 0$ in (4.11) and using (4.10), we obtain

$$\int_0^A \int_\Omega |\hat{y}|^2 dxda \le \int_0^A \int_\Omega |f|^2 dxda.$$

Then, $\big|\big|(i\beta I - \mathcal{B}_0)^{-1}\big|\big|$ is bounded and we conclude with the Lemma 4.1.

Remark 4.1. *The condition* (4.10) *implies that the basic reproduction number* $R_0 = \int_0^A \beta(a)e^{-\int_0^a \mu(s)ds}da$ *satisfies* $R_0 < 1$.

References

[1] Ó. Angulo, J. C. López-Marcos, M. Á. López-Marcos, J. Martínez-Rodríguez *An age-structured population model with delayed and space-limited recruitment,* Communications in Nonlinear Science and Numerical Simulation 112 (2022) 106545

[2] S. Anita, Analysis and control of age-dependent population dynamics. Kluwer Academic Publishers (2000).

[3] C. D. Benchimol, *A note on weak stabilization of contraction semigroups,* SIAM J. Control optim. 16(1978), 373 – 379.

[4] A. Borichev, Y. Tomilov, *Optimal polynmial decay of functions and operator semigroups,* Math. Ann. 347 (2010) 455–478.

[5] W. L. Chan, B. Z. Guo, *On the semigroups of age-size dependent population dynamics with spatial diffusion,* Manuscripta Math. 66(1989), $161-181$.

[6] K. J. Engel, R. Nagel, *One-Parameter Semigroups For Linear Evolution Equations,* Springer-Verlag New York, 2000.

[7] S. R. Foguel, *Powers of contraction in Hilert space,* Pacific J. Math., 13 ((196)), $551-561$.

[8] F. L. Huang, *Strong asymptotic stability of linear dynamical systems in Banach spaces,* J. Berlin, 1985

[9] K. Gu, J. Chan, V. L. Kharitonov, *Stability of time-delay systems,* Control Engineering, Birkhauser Boston, 2003

[10] M. Kumar, S. Abbas, A. Tridane ,*A novel method for basic reproduction ratio of a size-structured population model with delay,* Nonlinear Dynamics 109, 3189–3198 (2022)

[11] Q. Liu, Y. Jia, *Fluctuations-induced switch in the gene transcriptional regulatory system,* Phys. Rev. E 70, 041907 (2004).

[12] S. Ma, Q. Lu, S. Mei, *Dynamics of a logistic population model with maturation delay and nonlinear birth rate,* Discrete and Continuous Dynamical Systems-Series B, Vol. 5, Num. 3, pp. $735-752$

[13] F.M.G. Magpantay, N. Kosovalić, *An age-structured population model with state-dependent delay: Dynamics,* IFAC-PapersOnLine $48-12$ (2015) $099-104$

[14] K.Y. Ng, M.M. Gui, *Covid-19: development of a robust mathematical model and simulation package with consideration for ageing population and time delay for control action and resusceptibility,* Phys. D Nonlinear Phenomena 411, 132599 (2020).

[15] H. P. Oquendo, P. S. Pacheco, *Optimal decay for coupled waves with Kelvin-Voigt damping,* Applied Mathematics Letter 67 (2017).

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Applied Mathematical Sciences, Springer New York, (1983).

[17] J. Pruss, *On the Spectrum of $\mathcal{C}_0$-semigroups ,* Trans. Amer. Math. Soc. 284 (1984), $847-857$

[18] J. P. Richard, *Time delay systems: an overview of some recent advances and open problems,* Science direct automatica 39, $1667-1694$, 10 2003

[19] N. Shao, J. Cheng, W. Chen, *The reproductive number R_0 of COVID-19 based on estimate of a statistical time delay dynamical system,* (2020), https://doi.org/10.1101/2020.02.17.20023747.

[20] R. Silga, G. Bayili, *Polynomial stability of the wave equation with distributed delay term on the dynamical control,* Nonauton. Dyn. Syst. 2021; 8 : 207–227

[21] B. Sz-Nagy, C. Foias, *Analyse harmonique des opérateurs de l'espace de Hilbert,* Masson Paris, 1967.

[22] Y. Wang, X. Liu, Y. Wei, *Dynamics of a stage-structured single population model with state-dependent delay,* Advances in Difference Equations (2018) 2018 : 364

[23] D. Yan, *Long-time behavior of a size-structured population model with diffusion and delayed birth process,* Evolution Equations and Control Theory Vol. 11, No. 3, June 2022, pp. 895 – 923

[24] D. Zhang, H. Song, L. Yu, Q.-G. Wang; C.Ong, *Setvalues filtering for discrete time-delay genetic regulatory networks with time-varying parameters,* Nonlinear Dyn. 69, 693 – 703 (2012).