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Non-local boundary anisotropic problem with L^1-data and variable exponent

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Abstract : In this work, we study the following anisotropic problem
$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial}{\partial x_i}u)+\beta(u)\ni f$ in Ω, with non-local boundary conditions. We prove an existence and uniqueness of entropy solution for L^1-data f.

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1 Introduction

Our aim is to study the following problem

$$P(\rho,f,d)\ \begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial}{\partial x_i}u)+\beta(u)\ni f & \text{in } \Omega\\ u=0 & \text{on } \Gamma_D\\ \rho(u)+\sum_{i=1}^{N}\int_{\Gamma_{Ne}}a_i(x,\frac{\partial}{\partial x_i}u)\eta_i=d & \text{on } \Gamma_{Ne}\\ u\equiv constant & \text{on } \Gamma_{Ne}, \end{cases} \tag{1.1}$$

where, $d\in\mathbb{R}$ and ρ is a continuous and non decreasing function on $\mathbb{R}$, Ω is a open bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega=\Gamma_D\cup\Gamma_{Ne}$ with $\Gamma_D\cap\Gamma_{Ne}=\emptyset$ which means that Γ_D

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and Γ_{Ne} are partitions of the border $\partial\Omega$ of Ω. The right-hand side $f \in L^1(\Omega)$ and $\eta_i,\ i \in \{1, ..., N\}$ are the components of the outer unit normal vector. $\beta = \partial j$ is a maximal monotone graph in $\mathbb{R}$ with $dom(\beta)$ bounded on $\mathbb{R}$ such that $0 \in \beta(0)$.
For any $\Omega \in \mathbb{R}^N$, we set

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \inf_{x\in\Omega} h(x) > 1\}, \tag{1.2}$$

and we denote

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x). \tag{1.3}$$

For the exponents, $\vec{p}(.) : \bar{\Omega} \to \mathbb{R}^N$, $\vec{p}(.) = (p_1(.), ..., p_N(.))$ with $p_i \in C_+(\bar{\Omega})$ for every $i \in \{1, ..., N\}$ and for all $x \in \bar{\Omega}$. We put $p_M(x) = \max\{p_1(x), ..., p_N(x)\}$ and $p_m(x) = \min\{p_1(x), ..., p_N(x)\}$.
Note that j is a nonnegative, convex and l.s.c. function on $\mathbb{R}$ and, ∂j is the subdifferential of j.
We set

$$\overline{\mathrm{dom}(\beta)} = [m, M] \subset \mathbb{R} \text{ with } m \leq 0 \leq M \tag{1.4}$$

We assume that for $i = 1, ..., N$, the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory(i.e. $a_i(x, \xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}$) and satisfies the following conditions :

- (H_1) : There exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1(j_i(x) + |\xi|^{p_i(x)-1}), \tag{1.5}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(\cdot)}(\Omega)$, with $\dfrac{1}{p_i(x)} + \dfrac{1}{p_i'(x)} = 1$.
- (H_2) : There exists a positive constant C_2 such that

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1, \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1, \end{cases} \tag{1.6}$$

  for almost every $x \in \Omega$ and for every ξ, $\eta \in \mathbb{R}$, with $\xi \neq \eta$.
- (H_3) : For almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$,

$$|\xi|^{p_i(x)} \leq a_i(x, \xi)\xi. \tag{1.7}$$

- (H_4) : We also assume that the variable exponents $p_i(.) : \bar{\Omega} \to [2, N)$ are continuous functions for all $i = 1, ..., N$ such that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}},\ \sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \text{ and } \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{1.8}$$

  where $\dfrac{1}{\bar{p}} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \dfrac{1}{p_i^-}$.
- (H_5) : ρ is a continuous and non decreasing function on $\mathbb{R}$ such that

$$D(\rho) = Im(\rho) = \mathbb{R} \text{ and } \rho(0) = 0.$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1},\ q^* = \frac{Nq}{N-q} = \frac{N(\bar{p}-1)}{N-\bar{p}}. \tag{1.9}$$

The interest to study problems with variable exponents instead of constant exponent is linked to a large scale of applications that involve some nonhomogeneous materials (blood for example). It is already known that for an appropriate treatment of these materials, classical Sobolev and Lebesgue spaces are not adequate, so we have to allow the exponent to vary. We can refer here to electrorheological fluids (see [1, 9, 21]) thermorheological fluids, modelling of propagation of epidemic disease (see [3]), image restoration (see [8]). In order to answer to the preoccupation for the nonhomogeneous materials that behave differently on different spaces direction, the anisotropic space with variable exponents are introduced.

It is not a surprise to meet new difficulties when passing from isotropic variable exponent to anisotropic variables exponents. To overcome these difficulties, we combine the classical techniques with the recent techniques that have appeared when treating anisotropic problems with variables exponents.

The first systematic study of anisotropic Neumann problem was done by Fan (see [12]). After that, Boureanu et *al.* studied anisotropic nonhomogeneous Neumann problem with obstacle (see [6]). In the two papers, the authors were interested by the existence and multiplicity results of weak solutions even if in [6], Boureanu et *al.* have showed some conditions under which they can get uniqueness of weak solution.

All papers tackling the issues about (1.1) have considered particular cases (see [5, 14] and the references therein). The main interest in our work is that we are dealing with general non-linearities β which is a multivalued datum.

Regarding the border, in a recent paper, Kaboré and Ouaro used the technique of monotone operators in Banach spaces (see [17]) and approximation methods to get the existence and uniqueness of entropy solutions of the following problem,

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial}{\partial x_i} u) + |u|^{p_M(x)-2} u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \left.\begin{array}{l} \rho(u) + \sum_{i=1}^{N} \int_{\Gamma_{Ne}} a_i(x, \frac{\partial}{\partial x_i} u)\eta_i = d \\ u \equiv constant \end{array}\right\} & \text{on } \Gamma_{Ne}. \end{cases} \tag{1.10}$$

Our aim is to prove the existence and uniqueness of renormalized and entropy solutions to the general elliptic problem (1.1).

Non-local boundary value problems of various kinds for partial differential equations are of great interest by now in several fields of application. In a typical non-local problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the non-local behavior of u elsewhere in Ω. For example, at any point in Ω, the partial differential equation and/or the boundary conditions may contain integrals of the unknown u over parts of Ω, values of u elsewhere in D or, generally speaking, some non-local operator on u. Beside the mathematical interest of nonlocal conditions, it seems that this type of boundary condition appears in petroleum

engineering model for well modeling in a $3D$ stratified petroleum reservoir with arbitrary geometry (see [10] and [13]).
Since we assume that the domain of β is bounded, it appears in the definition of the solution, a bounded Radon diffuse measure in order to take into account the border of the domain. The techniques used in this work are close to those used in [15, 19].

The paper is organized as follows.In Section 2, we give some preliminaries about anisotropic Sobolev spaces of variable exponents and state our main result. Section 3, we study an approximated problem and in Section 4, we study the regularized problem corresponding to (1.1). In the Section 5, we prove the existence and uniqueness of entropy solution of problem (1.1) by using the results of the Section 4.

2 Preliminaries

This part is related to Lebesgue space and anisotropic Sobolev spaces of variable exponents and some of their properties.
Given a measurable function $p(.) : \Omega \to [1, \infty)$. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite.
If the exponent is bounded, i.e, if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf \left\{ \lambda > 0 : \rho_{p(.)}(\frac{u}{\lambda}) \leq 1 \right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Then, $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for all $x \in \Omega$.
Finally, we have the Hölder type inequality.

Proposition 2.1. *(see [11])*

(i) For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(.)} |v|_{p'(.)}.$$

(ii) If $p_1, p_2 \in C_+(\bar{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \bar{\Omega}$, then $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ and the imbedding is continuous.

We have the following properties on the modular $\rho_{p(.)}$.

Lemma 2.2 (see [11])**.** *If $u, u_n \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then*

$$|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^-}, \tag{2.1}$$

$$|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^+}, \tag{2.2}$$

$$|u|_{p(.)} < 1(= 1; > 1) \Rightarrow \rho_{p(.)}(u) < 1(= 1; > 1) \tag{2.3}$$

and

$$|u_n|_{p(.)} \to 0(\ p(.) \to \infty) \Leftrightarrow \rho_{p(.)}(u_n) \to 0(p(.) \to \infty). \tag{2.4}$$

We introduce the definition of the isotropic Sobolev space with variable exponent,

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measure in Ω, equipped with its standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$. Note that, if u belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure on Ω.
Given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $\mathrm{Cap}_{p(.)}(A, \Omega) = 0$.
For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A, u \geq 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset A with respect to Ω is defined by

$$\mathrm{Cap}_{p(.)}(A, \Omega) = \inf_{u \in S_{p(.)}(A)} \{ \int_\Omega |\nabla u|^{p(x)} dx \}.$$

In the case $S_{p(.)}(A) = \emptyset$, we set $\mathrm{Cap}_{p(.)}(A, \Omega) = \infty$.
The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$.
Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of (1.1).
The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(.)}(\Omega)$ is defined as follow.

$$W^{1,\vec{p}(.)}(\Omega) := \left\{ u \in L^{p_M(.)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega), \text{ for all } i \in \{1, ..., N\} \right\}.$$

Endowed with the norm

$$\|u\|_{\vec{p}(.)} := |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)},$$

the space $\left(W^{1,\vec{p}(.)}(\Omega), \|.\|_{\vec{p}(.)}\right)$ is a reflexive Banach space (see [12], Theorem 2.1 and Theorem 2.2). As consequence, we have the following.

Theorem 2.3. *(see [12]) Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded open set and for all $i \in \{1, ..., N\}$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e in Ω. Then, for any $r \in L^\infty(\Omega)$ with $r(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x \in \Omega} (p_M(x) - r(x)) > 0,$$

we have the compact embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega).$$

We also need the following trace theorem due to [6].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open set with smooth boundary and let $\vec{p}(.) \in C(\bar{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x \in \partial\Omega} \{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}, \ \forall x \in \partial\Omega, \tag{2.5}$$

where for all $x \in \partial\Omega$,

$$p_i^{\partial}(x) = \begin{cases} \dfrac{(N-1)p_i(x)}{N - p_i(x)} & \text{if } p_i(x) < N, \\ \infty & \text{if } p_i(x) \geq N. \end{cases}$$

Then, there is a compact boundary trace embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

Let us introduce the following notation :

$$\vec{p}_- = (p_1^-, ..., p_N^-).$$

Finally, in this paper, we will use the Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ with constant exponent. Note that the Marcinkiewicz spaces $\mathcal{M}^{q(.)}(\Omega)$ in the variable exponent setting was introduced for the first time by Sanchon and Urbano (see [22]).
Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ contain all measurable function $h : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_h(\gamma) := \text{meas}(\{x \in \Omega : |h(x)| > \gamma\}), \ \gamma \geq 0,$$

satisfies an estimate of the form $\lambda_h(\gamma) \leq C\gamma^{-q}$, for some finite constant $C > 0$.
The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|h\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{\frac{1}{q}} \left(\frac{1}{t} \int_0^t h^*(s)ds \right),$$

where h^* denotes the nonincreasing rearrangement of h :

$$h^*(t) := \inf \left\{ C : \lambda_h(\gamma) \leq C\gamma^{-q}, \ \forall \gamma > 0 \right\},$$

which is equivalent to the norm $\|h\|^*_{\mathcal{M}^q(\Omega)}$ (see [2]).
We need the following Lemma (see [4], Lemma A-2).

Lemma 2.5. *Let $1 \leq q < p < +\infty$. Then, for every measurable function u on Ω,*

(i) $\dfrac{(p-1)^p}{p^{p+1}} \|u\|^p_{\mathcal{M}^p(\Omega)} \leq \sup_{\lambda > 0} \{\lambda^p meas[x \in \Omega : |u| > \lambda]\} \leq \|u\|^p_{\mathcal{M}^p(\Omega)}$,

(ii) $\displaystyle\int_K |u|^q dx \leq \frac{p}{p-q} (\frac{p}{q})^{\frac{q}{p}} \|u\|^q_{\mathcal{M}^p(\Omega)} (meas(K))^{\frac{p-q}{p}}$, *for every measurable subset $K \subset \Omega$.*

In particular, $\mathcal{M}^p(\Omega) \subset L^q_{loc}(\Omega)$, with continuous injection and $u \in \mathcal{M}^p(\Omega)$ implies $|u|^q \in \mathcal{M}^{\frac{p}{q}}(\Omega)$.

We give some useful convergence results.

Lemma 2.6 (see [20]). *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of graphs (i.e. for $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$. We suppose that : $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then $z \in dom(\beta)$.*

The following result is due to Troisi (see [23]).

Theorem 2.7. *Let $p_1, ..., p_N \in [1, +\infty)$, $\vec{p} = (p_1, ..., p_N)$; $g \in W^{1,\vec{p}}(\Omega)$, and let*

$$\begin{cases} q = \bar{p}^* & if \quad \bar{p}^* < N, \\ q \in [1, +\infty) & if \quad \bar{p}^* \geq N; \end{cases} \tag{2.6}$$

where $p^* = \dfrac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1}$, $\sum_{i=1}^{N} \frac{1}{p_i} > 1$ *and* $\bar{p}^* = \dfrac{N\bar{p}}{N - \bar{p}}$.
Then, there exists a constant $C > 0$ depending on N, $p_1, ..., p_N$ if $\bar{p} < N$ and also on q and $meas(\Omega)$ if $\bar{p} \geq N$ such that

$$\|g\|_{L^q(\Omega)} \leq c \prod_{i=1}^{N} \left[\|g\|_{L^{p_M}(\Omega)} + \|\frac{\partial g}{\partial x_i}\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}, \tag{2.7}$$

where $p_M = \max\{p_1, ..., p_N\}$ and $\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$. In particular, if $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\|g\|_{L^q(\Omega)} \leq c \prod_{i=1}^{N} \left[\left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}. \tag{2.8}$$

In the sequel, we consider the following spaces (see [16, 17, 18]).

$$W_D^{1,\vec{p}(.)}(\Omega) = \{\xi \in W^{1,\vec{p}(.)}(\Omega) \ : \ \xi = 0 \text{ on } \Gamma_D\}$$

and

$$W_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \in W_D^{1,\vec{p}(.)}(\Omega) \ : \ \xi \equiv \text{constant on } \Gamma_{Ne}\}.$$

$$\mathcal{T}_D^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0, \ T_k(\xi) \in \ W_D^{1,\vec{p}(.)}(\Omega)\}$$

and

$$\mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0, \ T_k(\xi) \in \ W_{Ne}^{1,\vec{p}(.)}(\Omega)\},$$

where

$$T_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \leq k, \\ -k & \text{if } s < -k. \end{cases}$$

For any given $l, k > 0$, we define the function h_l by $h_l(r) = \min((l + 1 - |r|)^+, 1)$.
For any l_0, we consider the function $h_0 = h_{l_0}$ defined by

$$\begin{cases} h_0 \in C_c^1(\mathbb{R}), \ h_0(r) \geq 0, \ \forall r \in \mathbb{R}, \\ h_0(r) = 1 \text{ if } |r| \leq l_0 \text{ and } h_0(r) = 0 \text{ if } |r| \geq l_0 + 1. \end{cases} \tag{2.9}$$

For any $v \in W_{Ne}^{1,\vec{p}(.)}(\Omega)$, we set $v_{Ne} := v|_{\Gamma_{Ne}}$.
We recall the definition of the main section of a maximal monotone graph.
Let δ be a maximal monotone operator defined on $\mathbb{R}$. The main section δ_0 of δ is defined by

$$\delta_0(s) = \begin{cases} \text{the element of minimal absolute value of } \delta(s) & \text{if } \delta(s) \neq \emptyset, \\ \infty & \text{if } [s, \infty) \cap D(\delta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(\delta) = \emptyset. \end{cases}$$

We write for an $u : \Omega \to \mathbb{R}$ and $k \geq 0$, $[|u| \leq k(< k, > k, \geq k, = k)]$ for the set $\{x \in \Omega; |u(x)| \leq k(< k, > k, \geq k, = k)\}$.
The concept of solution for (1.1) is given as follows.

Definition 2.8. *A solution of* (1.1) *is a triple* $(u, w, v) \in \mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega) \times L^1(\Omega) \times \mathbb{R}$, $u \in dom(\beta)\mathcal{L}^N$*-a.e. in* Ω, $w \in \beta(u)\mathcal{L}^N$*-a.e. in* Ω, *there exists* $\mu \in \mathcal{M}_b^{p_m(.)}(\Omega)$ *with* $\mu \perp \mathcal{L}^N$, μ^+ *is concentrated on* $\{u = M\}$ *and* μ^- *is concentrated on* $\{u = m\}$, $v = \rho(u)$ *and*

$$\int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} \varphi \right) dx + \int_\Omega w\varphi dx + \int_\Omega \varphi d\mu = \int_\Omega f\varphi dx + (d - v)\varphi_{Ne}, \tag{2.10}$$

$\forall \varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.9. *If* (u, w, v) *is a solution of the problem* (1.1) *then, it satisfies the following entropic formulation*

$$\begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} T_k(u - \xi) \right) dx + \int_\Omega w T_k(u - \xi) dx \leq \\ \int_\Omega f T_k(u - \xi) dx + (d - v) T_k(u_{Ne} - \xi), \end{cases} \tag{2.11}$$

for all $\xi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$ *such that* $\xi \in dom(\beta)\mathcal{L}^N$*-a.e. in* Ω.

Our main result is the following.

Theorem 2.10. *Assume that* (1.4)-(1.8) *hold true and* $(f, d) \in L^1(\Omega) \times \mathbb{R}$, *there exists a unique entropy solution to problem* (1.1). *Moreover,*

$$\lim_{n \to +\infty} \int_{[n \leq |u| \leq n+1]} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx = 0. \tag{2.12}$$

Before proving Theorem 2.10, we study an auxiliary problem from which, we deduce useful a priori estimates.

3 Approximated problem for continuous functions

We define a new bounded domain $\tilde{\Omega}$ in $\mathbb{R}^N$ as follow (see Figure 1 below). We consider a fixed arbitrary $\theta > 0$, we consider the open bounded domain $\tilde{\Omega} \supset \Omega$, given by
$\tilde{\Omega} = \Omega \cup \{x \in \mathbb{R}^N / dist(x, \Gamma_{Ne}) < \theta\}$. Here, we consider $\theta > 0$ small enough such that $\partial\tilde{\Omega}$ is Lipschitz

and $\Gamma_D \subset \partial\tilde{\Omega}$. Then, let us denote by $\tilde{\Gamma}_{Ne} = \partial\tilde{\Omega} \setminus \Gamma_D$.

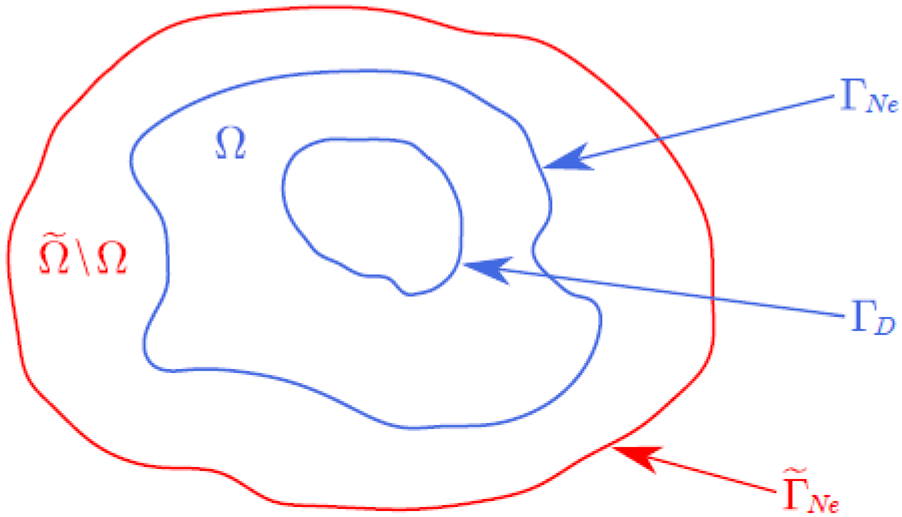

Figure 1: Domains representation

Let us consider $\tilde{a}_i(x,\xi)$ (to be defined later) Carathéodory and satisfying (1.5), (1.6) and (1.7), for all $x \in \tilde{\Omega}$.
We also consider a function $\tilde{d}$ in $L^1(\tilde{\Gamma}_{Ne})$ such that

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{d} d\sigma = d. \tag{3.1}$$

For any $\epsilon > 0$, we set $f_\epsilon = T_{\frac{1}{\epsilon}}(f)$ and $\tilde{f}_\epsilon = f_\epsilon \chi_\Omega$, $\tilde{d}_\epsilon = T_{\frac{1}{\epsilon}}(\tilde{d})$ and we consider the problem

$$P(\tilde{b},\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\tilde{a}_i(x,\frac{\partial}{\partial x_i}u_\epsilon) + \tilde{b}(u_\epsilon) = \tilde{f}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \displaystyle\sum_{i=1}^{N} \tilde{a}_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \tag{3.2}$$

where the functions $\tilde{b}$ and $\tilde{\rho}$ are defined as follow.

- $\tilde{b}(x,s) = b(s)\chi_\Omega(x)$, $\forall (x,s) \in \tilde{\Omega} \times \mathbb{R}$, where b is a continuous non-decreasing function such that $\mathcal{D}(b) = Im(b) = \mathbb{R}$ and $b(0) = 0$.
- $\tilde{\rho}(s) = \dfrac{1}{|\tilde{\Gamma}_{Ne}|}\rho(s)$, where $|\tilde{\Gamma}_{Ne}|$ denotes the Hausdorff measure of $\tilde{\Gamma}_{Ne}$.

We obviously have $\forall \epsilon > 0$, $\tilde{f}_\epsilon \in L^\infty(\tilde{\Omega})$, $\tilde{d}_\epsilon \in L^\infty(\tilde{\Gamma}_{Ne})$.
The following definition gives the notion of solution for the problem $P_\epsilon(\tilde{b},\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$.

Definition 3.1. *A measurable function* $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ *is a solution to problem* $P_\epsilon(\tilde{b}, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$ *if* $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ *and*

$$\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i} \tilde{\xi} dx + \int_{\Omega} b(u_\epsilon)\tilde{\xi} dx = \int_{\Omega} f_\epsilon \tilde{\xi} dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon))\tilde{\xi} d\sigma, \tag{3.3}$$

for any $\tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.

Thanks to Theorem 3.1 in [17], $P(\tilde{b}, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$ has at least one solution u and $|u| \leq C(b, k_1)$ a.e. in Ω and $|u| \leq C(\rho, k_2)$ a.e. on $\tilde{\Gamma}_{Ne}$ where k_1 and k_2 are defined as follow

$$\begin{cases} |b(u_{\epsilon,k})| \leq k_1 := \max\{\frac{\|f\|_1}{\text{meas}(\Omega)}; (b \circ \rho^{-1})(\|\tilde{d}\|_1)\} \text{ a.e. in } \Omega, \\ |\tilde{\rho}(u_{\epsilon,k})| \leq k_2 := \max\{\frac{\|\tilde{d}\|_1}{\text{meas}(\tilde{\Gamma}_{Ne})}; (\tilde{\rho} \circ b^{-1})(\frac{\|f\|_1}{\text{meas}(\Omega)})\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.4}$$

4 The regularized problem corresponding to (1.1)

For every $\epsilon > 0$, we consider the Yosida regularization β_ϵ of β given by

$$\beta_\epsilon = \frac{1}{\epsilon}(I - (I + \epsilon\beta)^{-1}),$$

and we set

$$j_\epsilon(s) = \min_{r \in \mathbb{R}} \left\{ \frac{1}{2\epsilon}|s - r|^2 + j(r) \right\}, \ \forall s \in \mathbb{R}, \forall \epsilon > 0.$$

According to Proposition 2.11 in [7], we have

$$\begin{cases} \text{dom}(\beta) \subset \text{dom}(j) \subset \overline{\text{dom}(j)} \subset \overline{\text{dom}(\beta)}. \\ j_\epsilon(s) = \dfrac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon), \text{ where } J_\epsilon = (I + \epsilon\beta)^{-1}, \\ j_\epsilon \text{ is convex, Frechet-differentiable and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon \downarrow 0. \end{cases}$$

Now, we set $\tilde{a}_i(x, \xi) = a_i(x, \xi)\chi_\Omega(x) + \dfrac{1}{\epsilon^{p_i(x)}}|\xi|^{p_i(x)-2}\xi\chi_{\tilde{\Omega}\setminus\Omega}(x)$ for all $(x, \xi) \in \tilde{\Omega} \times \mathbb{R}$, $\tilde{\beta}_\epsilon(x, s) = \beta_\epsilon(s)\chi_\Omega(x)$ for all $(x, s) \in \tilde{\Omega} \times \mathbb{R}$.
We consider the following problem denoted by $P_\epsilon(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a_i(x, \frac{\partial}{\partial x_i} u_\epsilon)\chi_\Omega(x) + \frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \chi_{\tilde{\Omega}\setminus\Omega}(x) \right) + \beta_\epsilon(u_\epsilon)\chi_\Omega = \tilde{f}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \displaystyle\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon)\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{4.1}$$

So, there exists at least one measurable function $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\int_{\Omega} a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}\tilde{\xi}dx + \sum_{i=1}^{N}\int_{\tilde{\Omega}\backslash\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}\tilde{\xi}\right)dx \\ \displaystyle + \int_{\Omega}\beta_\epsilon(u_\epsilon)\tilde{\xi}dx = \int_{\Omega} f_\epsilon\tilde{\xi}dx + \int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon))\tilde{\xi}d\sigma, \end{cases} \tag{4.2}$$

where $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and $\tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.
Moreover, by (3.4), we have

$$\begin{cases} |\beta_\epsilon(u_\epsilon)| \leq k_3 := \max\left\{\frac{\|f\|_1}{\text{meas}(\Omega)}; (\beta_\epsilon \circ \rho_0^{-1})(\|\tilde{d}\|_1)\right\} \text{ a.e. in } \Omega, \\ |\tilde{\rho}(u_\epsilon)| \leq k_4 := \max\left\{\frac{\|\tilde{d}\|_1}{\text{meas}(\tilde{\Gamma}_{Ne})}; (\tilde{\rho} \circ \beta_\epsilon^{-1})(\frac{\|f\|_1}{\text{meas}(\Omega)})\right\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{4.3}$$

The next result gives a priori estimates on the solution u_ϵ of the problem $P_\epsilon(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$.
Using the same argument as in [17], we have the following results.

Proposition 4.1. *Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then, the following statements hold.*

(i) $\forall k > 0$,

$$\sum_{i=1}^{N}\int_{\Omega}\left(|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|\right)^{p_i(x)}dx + \sum_{i=1}^{N}\int_{\tilde{\Omega}\backslash\Omega}\left(\frac{1}{\epsilon}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|\right)^{p_i(x)}dx \leq k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)});$$

(ii)

$$\int_{\Omega}|\beta_\epsilon(u_\epsilon)|dx + \int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u_\epsilon)|d\sigma \leq \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)};$$

(iii) $\forall k > 0$,

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx \leq k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}).$$

Lemma 4.2. *Assume* (1.4)-(1.8), *$f \in L^1(\Omega)$ and $\tilde{d} \in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then there is a positive constant D such that*

$$meas\{|u_\epsilon| > k\} \leq D^{p_m^-}\frac{(1+k)}{k^{p_m^- - 1}}, \ \forall k > 0.$$

Lemma 4.3. *Assume* (1.4)-(1.8), *$f \in L^1(\Omega)$ and $\tilde{d} \in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then there is a positive constant C such that*

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left(\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i^-}\right)dx \leq C(k+1), \ \forall k > 0. \tag{4.4}$$

Lemma 4.4. *Assume* (1.4)-(1.8), *$f \in L^1(\Omega)$ and $\tilde{d} \in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. For all $k > 0$, there is two constants C_1 and C_2 such that :*

(i) $\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \leq C_1$;

(ii) $\|\frac{\partial}{\partial x_i}u_\epsilon\|_{\mathcal{M}^{p_i^- q/p}(\tilde{\Omega})} \leq C_2$.

Proposition 4.5. *Assume* (1.4)-(1.8), $f \in L^1(\Omega)$ *and* $\tilde{d} \in L^1(\tilde{\Gamma}_{Ne})$. *Let* u_ϵ *be a solution of the problem* $P(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. *Then,*

(i) $T_k(u_\epsilon) \to T_k(u)$ *a.e. in* Ω *;*

(ii) $u_\epsilon \to u$ *in measure, a.e. in* Ω *and a.e. on* $\tilde{\Gamma}_{Ne}$ *;*

(iii) *For all* $i = 1, ...N$, $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} = 0$ *in* $L^{p_i(.)}(\tilde{\Omega} \setminus \Omega)$ *;*

(iv) *For all* $i = 1, ...N$, $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} = 0$ *in* $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.

Remark 4.6. *It is easy to see that* $u \in dom(\beta)$ *a.e. in* Ω. *Indeed, using Proposition 4.2-(i) and Lemma 2.3, we deduce that for all* $k > 0$, $T_k(u) \in dom(\beta)$ *a.e. in* Ω *and as* $dom(\beta)$ *is bounded, we deduce that* $u \in dom(\beta)$ *a.e. in* Ω.

Lemma 4.7. $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$.

Lemma 4.8. *Assume that* (1.4)-(1.8) *hold true and* u_ϵ *be a weak solution of the problem* $P(\tilde{\beta}_\epsilon, \tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. *Then,*

(i) $\frac{\partial}{\partial x_i}u_\epsilon$ *converges in measure to* $\frac{\partial}{\partial x_i}u$.

(ii) $a_i(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}) \to a_i(x, \frac{\partial T_k(u)}{\partial x_i})$ *strongly in* $L^1(\Omega)$ *and weakly in* $L^{p_i'(.)}(\Omega)$, *for all* $i = 1, ..., N$.

Proposition 4.9. *For any* $k > 0$ *and any* $i = 1, ..., N$, *as* ϵ *tends to* 0, *we have*

(i) $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to \frac{\partial T_k(u)}{\partial x_i}$ *a.e. in* Ω,

(ii) $a_i(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to a_i(x, \frac{\partial T_k(u)}{\partial x_i})\frac{\partial T_k(u)}{\partial x_i}$ *a.e. in* Ω *and strongly in* $L^1(\Omega)$,

(iii) $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to \frac{\partial T_k(u)}{\partial x_i}$ *strongly in* $L^{p_i(.)}(\Omega)$.

Lemma 4.10. *For any* $i = 1, ...N$, $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \to \frac{\partial}{\partial x_i}(h(u)\varphi) \text{ strongly in } L^{p_i(.)}(\Omega) \text{ as } \epsilon \to 0.$$

Proof. For any $i = 1, ...N$, $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{cases} \frac{\partial(h(u_\epsilon)\varphi)}{\partial x_i} - \frac{\partial(h(u)\varphi)}{\partial x_i} & = (h(u_\epsilon) - h(u))\frac{\partial \varphi}{\partial x_i} + h'(u_\epsilon)\varphi\left[\frac{\partial u_\epsilon}{\partial x_i} - \frac{\partial u}{\partial x_i}\right] + (h'(u_\epsilon) - h'(u))\varphi\frac{\partial u}{\partial x_i} \\ & := \Psi_1^\epsilon + \Psi_2^\epsilon + \Psi_3^\epsilon. \end{cases} \tag{4.5}$$

For the term Ψ_1^ϵ, we consider

$$\rho_{p_i(.)}(\Psi_1^\epsilon) = \int_\Omega \left|(h(u_\epsilon) - h(u))\frac{\partial \varphi}{\partial x_i}\right|^{p_i(x)} dx.$$

Set

$$\Theta_1^\epsilon(x) = \left|(h(u_\epsilon) - h(u))\frac{\partial \varphi}{\partial x_i}\right|^{p_i(x)}.$$

We have $\Theta_1^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and

$$|\Theta_1^\epsilon(x)| \leq C(h, p_i^-, p_i^+)\left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i(x)} \in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we get $\lim\limits_{\epsilon\to 0} \rho_{p_i(.)}(\Psi_1^\epsilon) = 0$.
Hence,

$$\|\Psi_1^\epsilon\|_{L^{p_i(.)}(\Omega)} \to 0, \text{ as } \epsilon \to 0. \tag{4.6}$$

For the term Ψ_2^ϵ, we consider

$$\rho_{p_i(.)}(\Psi_2^\epsilon) = \int_\Omega \left|h'(u_\epsilon)\varphi\left[\frac{\partial T_l(u_\epsilon)}{\partial x_i} - \frac{\partial T_l(u)}{\partial x_i}\right]\right|^{p_i(x)} dx,$$

for some $l > 0$ such that $\text{supp}(h) \subset [-l, l]$.
Set

$$\Theta_2^\epsilon(x) = \left|h'(u_\epsilon)\varphi\left[\frac{\partial T_l(u_\epsilon)}{\partial x_i} - \frac{\partial T_l(u)}{\partial x_i}\right]\right|^{p_i(x)}.$$

We have $\Theta_2^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and

$$|\Theta_2^\epsilon(x)| \leq C(h, p_i^-, p_i^+)\|\varphi\|_\infty \left|\frac{\partial T_l(u_\epsilon)}{\partial x_i} - \frac{\partial T_l(u)}{\partial x_i}\right|^{p_i(x)}.$$

Using Proposition 4.3 $-$ (iii), we get $\lim\limits_{\epsilon\to 0} \rho_{p_i(.)}\left(\frac{\partial T_l(u_\epsilon)}{\partial x_i} - \frac{\partial T_l(u)}{\partial x_i}\right) = 0$. Then,

$$\left|\frac{\partial T_l(u_\epsilon)}{\partial x_i} - \frac{\partial T_l(u)}{\partial x_i}\right|^{p_i(x)} \to 0 \text{ strongly in } L^1(\Omega).$$

The Lebesgue generalized convergence theorem allows to have

$$\lim_{\epsilon\to 0}\int_\Omega \Theta_2^\epsilon(x)dx = \lim_{\epsilon\to 0}\rho_{p_i(.)}(\Psi_2^\epsilon) = 0.$$

Hence,

$$\|\Psi_2^\epsilon\|_{L^{p_i(.)}(\Omega)} \to 0, \text{ as } \epsilon \to 0. \tag{4.7}$$

For the term Ψ_3^ϵ, we consider

$$\rho_{p_i(.)}(\Psi_3^\epsilon) = \int_\Omega \left|(h'(u_\epsilon) - h'(u))\varphi\frac{\partial T_l(u)}{\partial x_i}\right|^{p_i(x)} dx,$$

for some $l > 0$ such that $\text{supp}(h) \subset [-l, l]$.
Set

$$\Theta_3^\epsilon(x) = \left|(h'(u_\epsilon) - h'(u))\varphi \frac{\partial T_l(u)}{\partial x_i}\right|^{p_i(x)}.$$

We have $\Theta_3^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and

$$|\Theta_3^\epsilon(x)| \leq C(h, p_i^-, p_i^+, \|\varphi\|_\infty) \left|\frac{\partial T_l(u)}{\partial x_i}\right|^{p_i(x)} \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon \to 0} \int_\Omega \Theta_3^\epsilon(x) dx = \lim_{\epsilon \to 0} \rho_{p_i(.)}(\Psi_3^\epsilon) = 0.$$

Hence,

$$\|\Psi_3^\epsilon\|_{L^{p_i(.)}(\Omega)} \to 0, \text{ as } \epsilon \to 0. \tag{4.8}$$

Thanks to (4.6) − (4.8), we get

$$\|\Psi_1 + \Psi_2 + \Psi_3^\epsilon\|_{L^{p_i(.)}(\Omega)} \to 0, \text{ as } \epsilon \to 0,$$

and the lemma is proved. □

5 Existence and uniqueness of entropy solution

5.1 Existence of entropy solution

We are now able to prove the result of existence of entropy solution of the problem (1.1) announced in Theorem 2.10.

Proof. Thanks to Proposition 4.2 and as $\forall k > 0$, $\forall i = 1, ..., N$, $\dfrac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$, then $\forall k > 0$, $T_k(u) = \mathit{constant}\ a.e.$ on $\tilde{\Omega} \setminus \Omega$. Hence, we conclude that $u \in \mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega)$.
Let $\varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. We consider the function $\varphi_1 \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$ such that

$$\varphi_1 = \varphi \chi_\Omega + \varphi_{Ne} \chi_{\tilde{\Omega} \setminus \Omega};$$

we set $\tilde{\xi} = h_k(u_\epsilon)\varphi_1$ (see (2.9) for the definition of h_k by taking $l_0 = k$), in (4.2) to get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi)\right) dx \\ \displaystyle + \sum_{i=1}^N \int_{\tilde{\Omega} \setminus \Omega} \left(\frac{1}{\epsilon^{p_i(x)}} |\frac{\partial}{\partial x_i} u_\epsilon|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi_{Ne})\right) dx + \\ \displaystyle\int_\Omega \beta_\epsilon(u_\epsilon) h_k(u_\epsilon)\varphi dx = \int_\Omega f_\epsilon h_k(u_\epsilon)\varphi dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d} h_k(u_\epsilon)\varphi_{Ne} d\sigma - \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) h_k(u_\epsilon)\varphi_{Ne} d\sigma. \end{cases} \tag{5.1}$$

We need to pass to the limit in (5.1) as $\epsilon \to 0$. Note that

$$\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi)\right)dx=\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}T_k(u_\epsilon))\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi)\right)dx,$$

since $\operatorname{supp}(h_k)\subset[-k,k]$, then, by Lemma 4.5-(ii) and Lemma 4.10,

$$\lim_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}T_k(u_\epsilon))\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi)\right)dx=\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}T_k(u))\frac{\partial}{\partial x_i}(h_k(u)\varphi)\right)dx;$$

that is

$$\lim_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi)\right)dx=\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h_k(u)\varphi)\right)dx. \qquad (5.2)$$

For the second term in the left hand side of (5.1), we get

$$\lim_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi_{Ne})\right)dx=0. \qquad (5.3)$$

Indeed,

$$\begin{cases}\displaystyle\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi_{Ne})\right)dx=\\ \displaystyle\varphi_{Ne}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega\cap[|u_\epsilon|\le k]}\left(\frac{1}{\epsilon}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|\right)^{p_i(x)}h_k'(u_\epsilon)dx.\end{cases}$$

As $|u_\epsilon|\le k$, $h_k(u_\epsilon)=1$ and so $h_k'(u_\epsilon)=0$.
Therefore,

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}(h_k(u_\epsilon)\varphi_{Ne})\right)dx=0.$$

Hence, we get (5.3).
It is easy to see by the Lebesgue generalized convergence theorem that

$$\begin{cases}\displaystyle\lim_{\epsilon\to 0}\int_{\Omega}f_\epsilon h_k(u_\epsilon)\varphi dx=\int_{\Omega}fh_k(u)\varphi dx,\\ \displaystyle\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon h_k(u_\epsilon)\varphi_{Ne}d\sigma=\int_{\tilde{\Gamma}_{Ne}}\tilde{d}h_k(u)\varphi_{Ne}d\sigma.\end{cases} \qquad (5.4)$$

We know that $\forall k>0$, $T_k(u)=constant$ on $\tilde{\Omega}\setminus\Omega$, then, it yields that $u=constant=u_{Ne}$ on $\tilde{\Omega}\setminus\Omega$ and so on $\tilde{\Gamma}_{Ne}$. So, one has

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon h_k(u_\epsilon)\varphi_{Ne}d\sigma &= \int_{\tilde{\Gamma}_{Ne}}\tilde{d}h_k(u)\varphi_{Ne}d\sigma\\ &= h_k(u_{Ne})\varphi_{Ne}\int_{\tilde{\Gamma}_{Ne}}\tilde{d}d\sigma.\end{aligned}$$

Using (3.1),

$$\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon h_k(u_\epsilon)\varphi_{Ne}d\sigma = dh_k(u_{Ne})\varphi_{Ne}. \tag{5.5}$$

For the last term in (5.1), we have

$$\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)h_k(u_\epsilon)\varphi_{Ne}d\sigma = \varphi_{Ne}\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)h_k(u_\epsilon)d\sigma.$$

Since $\tilde{\rho}$ is non-decreasing and $\mathrm{supp}(h_k)\subset[-k,k]$,

$$|\tilde{\rho}(u_\epsilon)h_k(u_\epsilon)|\le \max\{\tilde{\rho}(-k),\tilde{\rho}(k)\}\in L^1(\tilde{\Gamma}_{Ne}).$$

By the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)h_k(u_\epsilon)\varphi_{Ne}d\sigma = \int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_{Ne})h_k(u_{Ne})\varphi_{Ne}d\sigma = \rho(u_{Ne})h_k(u_{Ne})\varphi_{Ne}. \tag{5.6}$$

Let us examine the last term in the left hand side of (5.1).
Since, for any $k>0$, $(h_k(u_\epsilon)\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k\in\mathcal{M}_b(\Omega)$, such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon)\rightharpoonup^* z_k,\ \text{in } \mathcal{M}_b(\Omega) \text{ as } \epsilon\to 0.$$

Moreover, for any $\varphi\in W^{1,\vec{p}(.)}_{Ne}(\Omega)\cap L^\infty(\Omega)$,

$$\begin{cases}\displaystyle\int_\Omega \varphi dz_k & \displaystyle = \int_\Omega f h_k(u)\varphi dx + dh_k(u_{Ne})\varphi_{Ne} - \rho(u_{Ne})h_k(u_{Ne})\varphi_{Ne} \\ & \displaystyle -\sum_{i=1}^N\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h_k(u)\varphi)\right)dx \\ & \displaystyle = \int_\Omega\left[f h_k(u)\varphi + \frac{1}{\mathrm{meas}(\Omega)}\left(dh_k(u_{Ne})\varphi_{Ne} - \rho(u_{Ne})h_k(u_{Ne})\varphi_{Ne}\right)\right]dx \\ & \displaystyle -\sum_{i=1}^N\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h_k(u)\varphi)\right)dx \\ & \displaystyle = \int_\Omega F h_k(u)\varphi dx - \sum_{i=1}^N\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h_k(u)\varphi)\right)dx,\end{cases}$$

where $F=\left(f-\frac{1}{\mathrm{meas}(\Omega)}d\chi_{\Gamma_{Ne}}\right)\in L^1(\Omega)$.
Therefore, $z_k\in\mathcal{M}_b^{p_m(.)}(\Omega)$ and for any $k\le l$,

$$z_k = z_l,\ \text{ on } [|T_k(u)|<k].$$

Let us consider the Radon measure μ defined by

$$\begin{cases} z = z_k, & \text{on } [|T_k(u)|<k] \text{ for } k\in\mathbb{N}^*, \\ z=0, & \text{on } \cap_{k\in\mathbb{N}^*}[|T_k(u)|=k]. \end{cases} \tag{5.7}$$

For any $h \in C_c^1(\mathbb{R}), h(u) \in L^\infty(\Omega, d|z|)$ and it is easy to see that for any $\varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{cases} \int_\Omega h(u)\varphi dz = \int_\Omega \left[fh(u)\varphi + \frac{1}{\text{meas}(\Omega)} \left(dh(u_{Ne})\varphi_{Ne} - \rho(u_{Ne})h(u_{Ne})\varphi_{Ne} \right) \right] dx \\ -\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h(u)\varphi) \right) dx. \end{cases} \tag{5.8}$$

Moreover, one has the following lemma.

Lemma 5.1 (see [20]). *The Radon-Nikodym decomposition of the measure z given by* (5.7) *with respect to $\mathcal{L}^N$,*

$$\nu = w\mathcal{L}^N + \mu, \quad \text{with } \mu \perp \mathcal{L}^N,$$

satisfies the following properties

$$\begin{cases} w \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \Omega, \ w \in L^1(\Omega), \ \mu \in \mathcal{M}_b^{p_m(.)}(\Omega), \\ \mu^+ \text{ is concentrated on } [u = M], \\ \text{and } \mu^- \text{is concentrated on } [u = m]. \end{cases}$$

To finish the proof of Theorem 2.10, we consider $\varphi_1 \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\varphi_1$ as test function in (4.2) to get

$$\begin{cases} \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \right) dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} |\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2} \frac{\partial}{\partial x_i}u_\epsilon \frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi_{Ne}) \right) dx \\ + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx = \int_\Omega f_\epsilon h(u_\epsilon)\varphi dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}h(u_\epsilon)\varphi_{Ne} d\sigma - \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)h(u_\epsilon)\varphi_{Ne} d\sigma. \end{cases} \tag{5.9}$$

By the Lebesgue generalized convergence theorem and as $u = constant$ on $\tilde{\Omega} \setminus \Omega$, it follows that

$$\begin{cases} \lim_{\epsilon\to 0} \int_\Omega f_\epsilon h(u_\epsilon)\varphi dx = \int_\Omega fh(u)\varphi dx, \\ \lim_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon h(u_\epsilon)\varphi_{Ne} d\sigma = dh(u_{Ne})\varphi_{Ne}. \end{cases} \tag{5.10}$$

The first term of (5.9) can be written as

$$\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}T_{l_0}(u_\epsilon))\frac{\partial}{\partial x_i}(h_0(u_\epsilon)\varphi) \right) dx,$$

for some $l_0 > 0$; then, by Lemma 4.5-(ii) and Lemma 4.6,

$$\begin{aligned} \lim_{\epsilon\to 0} \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \right) dx &= \lim_{\epsilon\to 0} \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}T_{l_0}(u_\epsilon))\frac{\partial}{\partial x_i}(h_0(u_\epsilon)\varphi) \right) dx \\ &= \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}T_{l_0}(u))\frac{\partial}{\partial x_i}(h_0(u)\varphi) \right) dx \\ &= \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h(u)\varphi) \right) dx. \end{aligned}$$

For the second term of (5.9), one has

$$\begin{cases}\displaystyle\lim_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi_{Ne})\right)dx = \\ \displaystyle\lim_{\epsilon\to 0}\sum_{i=1}^{N}\varphi_{Ne}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}|\frac{\partial}{\partial x_i}T_{l_0}(u_\epsilon)|\right)^{p_i(x)}h_0'(u_\epsilon)dx = 0.\end{cases}$$

For the last term of (5.9), one gets

$$\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)h(u_\epsilon)\varphi_{Ne}d\sigma = \rho(u_{Ne})h(u_{Ne})\varphi_{Ne} = vh(u_{Ne})\varphi_{Ne},$$

where $v = \rho(u_{Ne})$.
Thanks to the convergence results in Lemma 5.1 and Lemma 4.5 $-$ (ii), one gets from (5.9),

$$\begin{cases}\displaystyle\lim_{\epsilon\to 0}\int_\Omega\beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx = & \displaystyle\int_\Omega\left[fh(u)\varphi + \frac{1}{\text{meas}(\Omega)}\left(dh(u_{Ne})\varphi_{Ne} - \rho(u_{Ne})h(u_{Ne})\varphi_{Ne}\right)\right]dx \\ & \displaystyle -\sum_{i=1}^{N}\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h(u)\varphi)\right)dx \\ & \displaystyle = \int_\Omega h(u)\varphi d\nu = \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\mu.\end{cases}$$

Letting ϵ goes to 0 in (5.9), one obtains

$$\sum_{i=1}^{N}\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(h(u)\varphi)\right)dx+\int_\Omega h(u)w\varphi dx+\int_\Omega h(u)\varphi d\mu = \int_\Omega fh(u)\varphi dx+(d-v)h(u_{Ne})\varphi_{Ne}. \tag{5.11}$$

In (5.11), we take $h \in C_c^1(\mathbb{R})$ such that $[m,M] \subset supp(h) \subset [-l,l]$ and $h(s) = 1$ for all $s \in [-l,l]$. As $u \in \text{dom}(\beta)$, then $h(u) = 1$ and it yields that (u,w,v) is a solution of the problem (1.1).

5.2 Uniqueness of entropy solution

We are now ready to prove the result of uniqueness of the entropy solution of problem (1.1) announced in Theorem 2.10. Indeed, let (u_1,w_1,v_1) and (u_2,w_2,v_2) be two entropy solutions of (1.1). For (u_1,w_1,v_1), we take $\varphi = u_2$ as test function and for (u_2,w_2,v_2), we take $\varphi = u_1$ as test function in (2.11), to get for any $k > 0$,

$$\begin{cases}\displaystyle\int_\Omega\left(\sum_{i=1}^{N}a_i(x,\frac{\partial}{\partial x_i}u_1)\frac{\partial}{\partial x_i}T_k(u_1-u_2)\right)dx + \int_\Omega w_1T_k(u_1-u_2)dx \leq \\ \displaystyle\int_\Omega fT_k(u_1-u_2)dx + (d-v_1)T_k((u_1)_{Ne}-(u_2)_{Ne})\end{cases} \tag{5.12}$$

and

$$\begin{cases}\displaystyle\int_\Omega\left(\sum_{i=1}^{N}a_i(x,\frac{\partial}{\partial x_i}u_2)\frac{\partial}{\partial x_i}T_k(u_2-u_1)\right)dx + \int_\Omega w_2T_k(u_2-u_1)dx \leq \\ \displaystyle\int_\Omega fT_k(u_2-u_1)dx + (d-v_2)T_k((u_2)_{Ne}-(u_1)_{Ne}).\end{cases} \tag{5.13}$$

By adding (5.12) and (5.13), we obtain

$$\begin{cases} \displaystyle\int_\Omega \sum_{i=1}^N \left(a_i(x, \frac{\partial}{\partial x_i} u_1) - a_i(x, \frac{\partial}{\partial x_i} u_2) \right) \frac{\partial}{\partial x_i} T_k(u_1 - u_2) dx \\ \displaystyle + \int_\Omega (w_1 - w_2) T_k(u_1 - u_2) dx + (v_1 - v_2) T_k((u_2)_{Ne} - (u_1)_{Ne}) \leq 0. \end{cases} \tag{5.14}$$

For any $k > 0$ and from (5.14) it yields

$$\int_\Omega \sum_{i=1}^N \left(a_i(x, \frac{\partial}{\partial x_i} u_1) - a_i(x, \frac{\partial}{\partial x_i} u_2) \right) \frac{\partial}{\partial x_i} T_k(u_1 - u_2) dx = 0, \tag{5.15}$$

and

$$(v_1 - v_2) T_k((u_2)_{Ne} - (u_1)_{Ne}) = 0. \tag{5.16}$$

From $(5.15) - (5.16)$, it follows that there exists a constant c such that $u_1 - u_2 = c$ a.e. in Ω and $v_1 = v_2$. At last, let us see that $w_1 = w_2$ a.e. in Ω and $\mu_1 = \mu_2$. Indeed, for any $\varphi \in \mathcal{D}(\Omega)$, taking φ as test function in (2.10) for the solutions (u_1, w_1, v_1) and (u_2, w_2, v_2), after substraction, we get

$$\int_\Omega (w_1 - w_2) \varphi dx + \int_\Omega \varphi d(\mu_1 - \mu_2) = 0.$$

Hence,

$$\int_\Omega w_1 \varphi + \int_\Omega \varphi d\mu_1 = \int_\Omega w_2 \varphi dx + \int_\Omega \varphi \mu_2.$$

Therefore,

$$w_1 \mathcal{L}^N + \mu_1 = w_2 \mathcal{L}^N + \mu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get

$$w_1 = w_2 \text{ a.e. in } \Omega \text{ and } \mu_1 = \mu_2.$$

To end the proof of Theorem 2.10, we prove (2.12).
We take $\tilde{\xi} = T_1(u_\epsilon - T_n(u_\epsilon))$ as test function in (2.10) to get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i} T_1(u_\epsilon - T_n(u_\epsilon)) dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \frac{\partial}{\partial x_i} T_1(u_\epsilon - T_n(u_\epsilon)) \right) dx \\ \displaystyle + \int_\Omega \beta_\epsilon(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon)) dx = \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon)) dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) T_1(u_\epsilon - T_n(u_\epsilon)) d\sigma. \end{cases} \tag{5.17}$$

Since $\displaystyle\int_\Omega \beta_\epsilon(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon)) dx \geq 0$, $\displaystyle\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon)) d\sigma \geq 0$ and

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \frac{\partial}{\partial x_i} T_1(u_\epsilon - T_n(u_\epsilon)) \right) dx = \\ \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega \cap [n \leq |u_\epsilon| \leq n+1]} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)} \right) dx \geq 0, \end{cases}$$

from equality (5.17), it follows that

$$\sum_{i=1}^{N}\int_{[n\leq|u_\epsilon|\leq n+1]} a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}u_\epsilon dx \leq \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))d\sigma. \quad (5.18)$$

By using the Lebesgue generalized convergence theorem and the Lebesgue dominated convergence theorem respectively, we prove that

$$\lim_{n\to 0}\lim_{\epsilon\to 0}\int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx = 0 \quad (5.19)$$

and

$$\lim_{n\to 0}\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))d\sigma = 0. \quad (5.20)$$

Using (1.7), it follows by letting $\epsilon \to 0$ and $n \to 0$ respectively in (5.17),

$$\lim_{n\to 0}\lim_{\epsilon\to 0}\sum_{i=1}^{N}\int_{[n\leq|u_\epsilon|\leq n+1]}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)} dx = \lim_{n\to 0}\sum_{i=1}^{N}\int_{[n\leq|u|\leq n+1]}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \leq 0. \quad (5.21)$$

Therefore, we get (2.12). □

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