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SOMMAIRE

Asymptotic Equivalence of OLS (GLS) and Maximum Likelihood using Cointegrated Systems with Higher Order Integrated Variables

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Abstract

This paper is concerned with the estimation of cointegrated systems with integrated variables of order greater than 1. Unlike in the case of order 1 cointegrated variables I(1), there are various possibilities of cointegration in the higher order case, which were conveniently formulated in a triangular representation and estimated by Ordinary Least Squares (OLS) and Generalized Least Squares (GLS) by Stock and Watson (1993). Starting from this triangular representation, we derive an error correction model that already incorporates the different cointegration restrictions and apply maximum likelihood to estimate the parameters. Our approach is compared with that of Johansen (1995) and Kitamura (1995). Asymptotic properties of our maximum likelihood (ML) estimators are derived. Further, it is shown that as far as the coefficients of integrated variables are concerned, our ML estimators are asymptotically equivalent to the OLS/GLS estimators of Stock and Watson (1993).

Keywords : Cointegration, Triangular Representation, Error Correction Model, Asymptotic equivalence.

1. Introduction

The concepts of nonstationary and cointegration have now become a fundamental component of an econometrician's stock of knowledge and error correction models combining short run dynamics with long term equilibria are frequently estimated in various situations. The pioneering article by Engle and Granger (1987) has been the real starting point of such research by economists though the basic notions have been known to statisticians even before (cf. Hsiao (1997)). The main motivation behind the increasing interest in this area is the practical observation that many economic time series are in fact nonstationary. Hence in order to maintain a theoretical relationship among these series in levels, which often has a better economic meaning than that in differences, these series need to be cointegrated such that the linear combination is stationary.

There is a vast literature on estimation and inference in cointegrated systems with I(1) variables . Before recently, many studies have focused on systems with higher order integrated variables (cf. Johansen (1992 and 1995), [1]Stock and Watson (1993), Kitamura (1995)). The empirical usefulness of such works has been illustrated by findings (eg. King, Plosser, Stock & Watson (1991)) that certain macroeconomic variables such as prices (in nominal terms) are in fact integrated of order 2.

One of the fundamental differences between I(1) and I(2) or higher order systems is that there are several possible combinations for cointegration. For instance if prices are 1(2) variables then either different prices can combine to give an I(1) series which in turn may cointegrate with inflation (which will be I(1)) to become stationary or the different prices may directly produce a stationary series by cointegration. This necessitates appropriate modelling that includes the different possibilities. Stock and Watson (1993) propose a triangular representation incorporating the different cointegrations, extending the one proposed by Phillips (1991) in

[1] see for instance Phillips and Durlauf (1986), Engle and Granger (1987), Engle and Yoo (1987), Stock (1987), Johansen (1988), Phillips (1988), Park and Phillips (1988, 1989), Johansen and Juselius (1990), Phillip s and Hansen (1990), Phillips and Ouliaris (1990), Sims, Stock and Watson (1990), Phillips (1991)

the I(1) case to the general case of a vector of variables with maximum order of integration d and discuss the estimation of the model by least squares methods and examine the asymptotic and small sample properties.

In this paper, we take their representation as a starting point to derive an appropriate error correct ion model and apply maximum likelihood to estimate the parameters. Haldrup and Salmon (1998) present a similar model while discussing various representations of I(2) systems using the Smith-McMillan form without going further to the estimation stage. We also differ from both Johansen (1995) and Kitamura (1995) in that the former does not incorporate the cointegration restrictions in the specification of the model but rather imposes it later and the latter approaches the maximum likelihood through decomposition into conditional likelihoods.

We propose a more direct approach as will be seen later. The rest of the paper is organisated as follows. In Section 2, we present the model and its error correction representation. The maximum likelihood method is described in Section 3 in which it is also compared with Johansen's (1995) and Kitamura's (1995) approaches.

Section 4 derives the limiting distribution of these estimators and Section 5 establishes the asymptotic equivalence of the least squares of Stock and Watson (1993) and our ML estimation for the coefficients of nonstationary variables. The paper ends with a concluding note in section 6.

2. The Model

Our starting point for studying higher order (co)integrated systems is the triangular representation of Stock and Watson (1993). This type of representation has also been used by other authors for the I(l) model (see Phillips and Hansen (1990), Phillips (1991) and Saikkonen (1991)). Let us consider a stochastic process of order n, y_t , with the maximum order of integration d. The Wold representation of $\triangle^d y_t$ is given by:

$$\triangle^d y_t = F(L)\varepsilon_t \quad (1)$$

with ε_t a white noise, $F(L) = \sum_{i=0}^{\infty} f_i L^i$ a lag polynomial, $\triangle^d = (1-L)^d$ and L being the lag operator. The triangular representation of $\{y_t\}_t$ *Stock & Watson (1993)* is written as follows:

$$\triangle^d y_{1t} = u_{1t}$$

$$\triangle^{d-1} y_{2t} = \theta_{2,1,d-1} \triangle^{d-1} y_{1t} + u_{2t}$$

.

.

$$\triangle^{d-l+1} y_{lt} = \sum_{j=1}^{l-1}\sum_{i=j}^{l-1} \theta_{l,j,d-i} \triangle^{d-i} y_{jt} + u_{lt} \qquad (2)$$

.

.

$$y_{d+1,t} = \sum_{j=1}^{d}\sum_{i=j}^{d} \theta_{d+1,j,d-i} \triangle^{d-i} y_{jt} + u_{d+1,t}$$

where

$$u_{it} = G_i(L)\varepsilon_t \ for \ i \in \{1,2,3,\dots,d+1\}$$

With $G_i(L)$ a matrix lag polynomial, $G_i(1) < \infty$ and for all $z \neq 0$, the matrix $G(z)$ is non-singular and $k-d$ summable for as defined in Assumption 4.

Note that

1. The triangular representation splits y_t into stochastic trends of different orders leading to partition of y_t as:

   $y_t' = [y_{1t}', y_{2t}', \dots, y_{lt}', \dots, y_{dt}', y_{d+1,t}']$

   where y_{lt} is $k_l \, x \, 1$ with an order of integration less than or equal to d.

2. $F(L)$ be partitioned in the same way:

   $F'(L) = [F_1'(L), F_2'(L), \dots, F_l'(L), \dots, F_d'(L), F_{d+1}'(L)]$

   similarly $u_t' = [u_{1t}', u_{2t}', \dots, u_{lt}', \dots, u_{dt}', u_{d+1,t}']$

By construction, there is a contemporaneous correlation among the u_{st}:
$E(u_{pt}u_{qt}') \neq 0$ for $p, q \in \{1,2, \dots, d, d+1\}$. Thus $E(\triangle^{d-i} y_{jt}u_{lt}') \neq 0$ for all
$j \in \{1,2, \dots, l-1\}$, $i \in \{j, j+1, \dots, l+1\}$ *and* $l \in \{2,3, \dots, d, d+1\}$.

Hence there is non-zero correlation between the explanatory variables and the
errors of each equation. *Stock and Watson (1993)* overcome this problem by using orthogonal projections:

$v_{lt} = u_{lt} - proj(u_{lt}/u_{1t}, \dots, u_{l-1,t})$, $l \in \{2,3, \dots, d, d+1\}$, where

$$proj\left(u_{lt}/u_{1t},\dots,u_{l-1,t}\right) = \sum_{m=1}^{l-1} D_{lm}(L)\left[\triangle^{d-m+1} y_{mt} - \sum_{j=1}^{m-1} \theta_{mj,d-i} \triangle^{d-i} y_{jt}\right]$$

is obtained by projecting u_{lt} onto leads and lags of $\{u_{1t},\dots,u_{l-1,t}\}$.

Substituting the projection in the l^{th} equation of the system (2), we obtain

$$\triangle^{d-l+1} y_{lt} = \sum_{j=1}^{l-1}\sum_{i=j}^{l-1} \theta_{lj,d-i} \triangle^{d-i} y_{jt} + \sum_{m=1}^{l-1} D_{mt}(L)\left[\triangle^{d-m+1} y_{mt} - \sum_{j=1}^{m-1}\sum_{i=j}^{m-1} \theta_{mj,d-i} \triangle^{d-i} y_{jt}\right] + v_{lt} \quad (3)$$

with v_{lt} and $\triangle^{d-i} y_{jt}$ being orthogonal for $j \in \{1,2,\dots,l-1\}$; $i \in \{j,\dots,l-1\}$
and $D_{lm}(L) = \sum_{j=1}^{q} d_{lm,j} L^j$ for $q < \infty$ and $m \in \{1,\dots,l-1\}$.

After a final canonical transformation (see *Sims, Stock and Watson (1992)),*
the system is rewritten in the following way:

$$\triangle^{d-l+1}\, y_{lt} = \left(x_t' \otimes I_{k_l}\right)\beta_l + v_{it} \qquad (4)$$

with $l = 1,2, \dots, d+1$. The regressors in x_t are linear combinations of $(y_{it}, \triangle\, y_{it}, \dots, \triangle^{d-l}\, y_{it})$, $i \in \{1,2, \dots, d-l\}$ with fixed coefficients and partitioned as $(x_{0t}, x_{1t}, \dots, x_{dt})$ such that $x_{kt} \sim I(k)$. β is the unconstrained coefficient vector composed
of $\theta_{lj,d-i}$; $j \in \{1, \dots, l-1\}$ and $d_{lm,j}, m \in \{1, \dots, l-1,\}, j \in \{1, \dots, q\}$. Let us recall the contemporaneous non-correlation among v_{lt} for $l \in \{1,2,3, \dots, d+1\}$ where $v_{1t} = u_{1t}$.
Since the errors in (4) are not correlated with the regressors, either ordinary least squares or generalized least squares can be applied to obtain consistent estimates of the unknown parameters in β. We will not go into a detailed discussion of these estimators, referring the reader to the article by *Stock and Watson (1993).*

We will show later that both the OLS and GLS estimators of the coefficients of variables integrated of order greater than or equal to 1 are asymptotically equivalent to the maximum likelihood estimator developed below.

The method consist in first to transform the triangular representation into an error correction model and then apply maximum likelihood to it. The procedure will be studied for the case in which the maximum order of integration is 2; however, the approach can be generalized without much difficulty to higher orders.

3. Estimation by the maximum likelihood method

Let us partition y_t (with maximum order of integration 2) as $y_t' = (y_{1t}', y_{2t}', y_{3t}')$
with $y_{it}\,(n_i \times 1)$ and $\sum_{i=1}^{3} n_i = n$. For each $i \in \{1,2,3\}$, we have $\triangle^2 y_{it} = F_i(L)\epsilon_t$
The *Stock and Watson (1993)* triangular representation for y_t is

$$\begin{cases} \triangle^2 y_{1t} = u_{1t} & (1) \\ \triangle y_{2t} = \theta_1 \triangle y_{1t} + u_{2t} & (2) \\ y_{3t} = \theta_2 y_{2t} + \theta_3 y_{1t} + \theta_4 \triangle y_{1t} + u_{3t} & (3) \end{cases} \qquad (5)$$

Where $u_t' = (u_{1t}', u_{2t}', u_{3t}')$ and $u_{it} = G_i(L)\epsilon_t$ with u_{it} , $n_i \times 1$ for $i = 1,2,3$

Note that the first equation signifies that y_{1t} is $I(2)$. Regarding the second one, we see that if θ_1 is zero then $\triangle y_{2t} = u_{2t}$ and hence y_{2t} is $I(1)$ whereas if θ_1 is not zero then y_{2t} is $I(2)$, y_{1t} and y_{2t} cointegrate to produce an $I(1)$series, written as $(y_{1t}, y_{2t}) \sim CI(2,1)$. In the first case where y_{2t} is $I(1)$, equation (3) indicates that (y_{1t}, y_{2t}) is $CI(2,1)$ and y_{2t} , $\triangle y_{1t}$ and the couple (y_{3t}, y_{1t}) are $CI(1,1)$. When θ_1 is not zero, two interpretations are possible for the third equation.

Either $(y_{3t}, y_{2t}, y_{1t}) \sim CI(2,1)$ and $[(y_{3t}, y_{2t}, y_{1t}), \triangle y_{1t}] \sim CI(1,1)$. But $(y_{2t}, y_{1t}) \sim CI(2,1)$ and y_{3t} , $\triangle y_{1t}$, the couple (y_{2t}, y_{1t}) are $CI(1,1)$. Note that if $\theta_4 = 0$ (when θ_1 is not zero), then the third equation implies that $(y_{3t}, y_{2t}, y_{1t}) \sim CI(2,2)$. [1]

We have $F(L)' = [F_1(L)', F_2(L)', F_3(L)']$. In order for the second equation of (5) to hold when θ_1 is not zero in other words when $\triangle y_{2t}$ and $\triangle y_{1t}$ are $CI(1,1)$ it is necessary that $F_2(1) = \theta_1 F_1(1)$. This implies that the columns of $F_2(1)$ are linear combinations of those of $F_1(1)$ (*see Stock & Watson* (1993)).

Using the orthogonal projections to eliminate correlations between regressors and errors the model becomes:

$$\triangle^2 y_{1t} = v_{1t}$$

$$\triangle y_{2t} = D_1(L)\triangle^2 y_{1t} + \theta_1 \triangle y_{1t} + v_{2t} \qquad (6)$$

$$y_{3t} = D_2(L)\triangle^2 y_{1t} + D_3(L)(\triangle y_{2t} - \theta_1 \triangle y_{1t}) + \theta_2 y_{2t} + \theta_3 y_{1t} + \theta_4 \triangle y_{1t} + v_{3t}$$

where v_{1t} , v_{2t} and v_{3t} are non-correlated by construction.

Recall that $v_{1t} = u_{1t}$; $v_{2t} = u_{2t} - proj\left(u_{2t}/\{u_{1,t-j}\}_{j=0,1,2,\ldots}\right)$ and $v_{3t} = u_{3t} - proj\left(u_{3t}/\{u_{1,t-j}, u_{2,t-j}\}_{j=0,1,2,\ldots}\right)$ where $proj\left(u_{2t}/\{u_{1,t-j_j}\right) = D_1(L)u_{1t}$ and $proj\left(u_{3t}/\{u_{1,t-j}, u_{2,t-j}\}_j\right) = D_2(L)u_{1t} + D_3(L)u_{2t}$
are respectively the orthogonal projections of u_{2t} on the space generated by $\{u_{1,t-j}\}_{j=0,1,2,\ldots}$ and of u_{3t} on the space generated by $\{u_{1,t-j}\}_{j=0,1,2,\ldots}$and $\{u_{2,t-j}\}_{j=0,1,2,\ldots}$.

Now, we transform the above system into an error correction model. In Annex 1, we describe the derivation procedure[1] which leads to the following error correction model:

$$\triangle^2 y_t = A(L)\triangle^2 y_{t-1} + J\Theta' x_{t-1} + w_t \qquad (7)$$

Which can be rewritten as:

$$\triangle^2 y_t = A(L) \triangle^2 y_{t-1} + B \triangle y_{t-1} + C y_{t-2} + w_t \qquad (8)$$

With

$$A(L) = \begin{pmatrix} 0 & 0 & 0 \\ A_1(L) & 0 & 0 \\ A_2(L) & A_3(L) & 0 \end{pmatrix}$$; polynomial of order p.

$$\mathrm{J} = \begin{pmatrix} 0 & 0 & 0 \\ I_{n_2} & 0 & 0 \\ 0 & I_{n_3} & I_{n_3} \end{pmatrix};$$

$$\Theta' = \begin{pmatrix} \theta_1 & -I_{n_2} & 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & -2I_{n_3} & 0 & 0 & 0 \\ 0 & 0 & \theta_3 & \theta_3 & \theta_2 & -I_{n_3} \end{pmatrix};$$

$$x'_{t-1} = (\triangle y'_{t-1} \quad y'_{t-2}) \ ;$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ \theta_1 & -I_{n_2} & 0 \\ \delta_1 & \delta_2 & -2I_{n_2} \end{pmatrix} \ ; \ rank(B) = n_2 + n_3 < n$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -I_{n_2} & 0 \\ \theta_3 & \theta_2 & -I_{n_3} \end{pmatrix} \ ; \ rank(C) = n_3 < n$$

$\triangle^2 y_t' = (\triangle^2 y_{1t}' , \ \triangle^2 y_{2t}' , \ \triangle^2 y_{3t}');$
$\triangle^2 y_{t-1}' = \left(\triangle^2 y_{1,t-1}' , \ \triangle^2 y_{2,t-1}' , \ \triangle^2 y_{3,t-1}'\right)$

$\triangle y_{t-1}' = \left(\triangle y_{1,t-1}' , \ \triangle y_{2,t-1}' , \ \triangle y_{3,t-1}'\right);$
$\triangle y_{t-2}' = \left(y_{1,t-2}' , \ y_{2,t-2}' , \ y_{3,t-2}'\right)$

and $w_t' = (w_{1t}', \ w_{2t}', \ w_{3t}')$.

At this point, let us note that our model is very similar to that of *Johansen* (*1995*)
with the following correspondence between the two notations:
$A(L) \equiv \Lambda(L)$, $\Gamma \equiv B$, $\Pi \equiv C$.
However, our approach differs both from the point of view of specification of the model and that of the method of estimation. The first difference is a fundamental one. By starting from the triangular representation, we have already incorporated the cointegration restrictions in our model whereas *Johansen's* (*1995*) model specification is a general one where the same restrictions must be imposed.

We will discuss the other point concerning the estimation method after presenting our method below. Before writing the likelihood function, we will rewrite our model (8) in a more convenient form. Let us note that:

- $A(L) \triangle^2 y_{t-1} = A\xi_{t-1}' = (\xi_{t-1}' \otimes I_n)vec(A)$

  with $A(n \times np) = [A_1 \ \ A_2 \ \dots \ A_{np}]$

  and $\xi_{t-1}'(1 \times np) = \left[\triangle^2 y_{t-1}' \ \ \triangle^2 y_{t-2}' \ \dots \ \triangle^2 y_{t-p}'\right]$

- $B \triangle y_{t-1} = (\triangle y_{t-1}' \otimes I_n)vec(B)$

- $C y_{t-2} = (y_{t-2}' \otimes I_n)vec(C)$

Substituting these results in (8), we obtain:

$$\triangle^2 y_t = (\xi'_{t-1} \otimes I_n)vec(A) + (\triangle y'_{t-1} \otimes I_n)vec(B) + (y'_{t-2} \otimes I_n)vec(C) + w_t \qquad (9)$$

Suppose we have T observations. Then piling them together, we can write:

$$vec(\triangle^2 Y') = (\xi_{-1} \otimes I_n)vec(A) + (\triangle Y_{-1} \otimes I_n)vec(B) + (Y_{-2} \otimes I_n)vec(C) + vec(W') \quad (10)$$

where $\triangle^2 Y'(n \times T) = [\triangle^2 y_1 \ \ \triangle^2 y_2 \ \ .... \ \triangle y_T]$;
$\xi'_{-1}(np \times T) = [\xi_0 \ \ \xi_1 \ \ ... \ \ \xi_{T-1}]$

$\triangle Y'_{-1}(n \times T) = [\triangle y_0 \ \ \triangle y_1 \ \ ... \ \ \triangle y_{T-1}]$;
$Y'_{-2}(n \times T) = [y_{-1} \ \ y_0 \ \ .... \ \ y_{T-2}]$

$W'(n \times T) = [w_1 \ \ w_2 \ \ .... \ \ w_T]$

Assumption 1:

The errors w_t are independently and normally distributed as follows:

$w_t = NID(0,\ \Omega)$

Thus

$w = vec(W') \sim N(0\ ,\ I_T \otimes \Omega)$

With this assumption, the log-likelihood is given by:

$$L(vec(A), vec(B), vec(C), \Omega) = -\frac{Tn}{2} ln(2\pi) + \frac{T}{2} ln\left|\Omega^{-1}\right| - \frac{1}{2} w'(I_T \otimes \Omega)^{-1} w \qquad (11)$$

$$= -\frac{Tn}{2} ln(2\pi) + \frac{T}{2} ln\left|\Omega^{-1}\right|$$

$$-\frac{1}{2}[\triangle^2 y_t - (\xi'_{t-1} \otimes I_n)vec(A) - (\triangle y'_{t-1} \otimes I_n)vec(B) - (y'_{t-2} \otimes I_n)vec(C)]'$$

$$(I_T \otimes \Omega)\ [\triangle^2 y_t - (\xi'_{t-1} \otimes I_n)vec(A) - (\triangle y'_{t-1} \otimes I_n)vec(B) - (y'_{t-2} \otimes I_n)vec(C)]$$

The procedure of maximizing *L* with respect to *A, B, C* and Ω will be decomposed into two steps. In the first step, we fix *B, C,* Ω and maximize with respect to *A*.
Doing this result s in a *SUR* model of *Johansen* (*1988*) for the elements of
$\triangle^2 y_t - (\triangle y'_{t-1} \otimes I_n)vec(B) - (y'_{t-2} \otimes I_n)vec(C)$ regressed on the explanatory variables ξ_{t-1}. Hence, $vec(A)$ can be estimated by ordinary least squares. In the second stage, we substitute $vec(A)$ by its estimator $vec(\hat{A})$ in the likelihood function and maximize the concentrated likelihood with respect to the other parameters.
Formally, in the first stage we get:

$$vec(\hat{A}) = [(\xi'_{-1} \otimes I_n)(\xi_{-1} \otimes I_n)]^{-1}(\xi'_{-1} \otimes I_n)$$
$$[vec(\triangle^2 Y') - (\triangle Y_{-1} \otimes I_n)vec(B) - (Y_{-2} \otimes I_n)vec(C)]$$
$$=[(\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1} \otimes I_n]vec(\triangle^2 Y') - [((\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1} \triangle Y_{-1} \otimes I_n]vec(B)$$
$$- [(\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1} \triangle Y_{-2} \otimes I_n]vec(C) \quad (12)$$

In the second stage, substituting (12) in (10) and rearranging yields:

$$vec(\triangle^2 Y') - [\xi_{-1}(\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1} \otimes I_n]vec(\triangle^2 Y')$$

$$= [\triangle Y_{-1} \otimes I_n - \xi_{-1}(\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1} \triangle Y_{-1} \otimes I_n]vec(B)$$

$$+ [Y_{-2} \otimes I_n - \xi_{-1}(\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1}Y_{-2} \otimes I_n]vec(C) + vec(W')$$

$$\Rightarrow (M_\xi \otimes I_n)vec(\triangle^2 Y') = (M_\xi \triangle Y_{-1} \otimes I_n)vec(B) + (M_\xi Y_{-2} \otimes I_n)vec(C) + vec(W')$$

$$\Leftrightarrow x = X_1 vec(B) + X_2 vec(C) + w \quad (13)$$

where $x = (M_\xi \otimes I_n)vec(\triangle^2 Y')$, $X_1 = M_\xi \triangle Y_{-1} \otimes I_n$, $X_2 = M_\xi Y_{-2} \otimes I_n$
with $M_\xi = I_T - \xi_{-1}(\xi'_{-1}\xi_{-1})^{-1}\xi'_{-1}$.

Here it is useful to take into account any restrictions on the parameters *B* and *C* by writing them as (*see Granger* (*1987*)):

$$vec(B) = q_1 + R_1\Pi_1 \quad (14)$$

where $q_1(n^2 \times 1)$; $R_1(n^2 \times k_1)$; $\Pi_1(k_1 \times 1)$

$$vec(C) = q_2 + R_2\Pi_2 \quad (15)$$

where $q_2(n^2 \times 1)$; $R_2(n^2 \times k_2)$; $\Pi_2(k_2 \times 1)$

(q_1 , q_2) and (R_1 , R_2) are known vectors and matrices respectively and (Π_1 , Π_2) are unknown vectors to be estimated.
Inserting (14) and (15) in (13), the model becomes

$$x = X_1(q_1 + R_1\Pi_1) + X_2(q_2 + R_2\Pi_2) + w$$

or

$$z = Z_1\Pi_1 + Z_2\Pi_2 + w \quad (16)$$

where $z = x - X_1q_1 - X_2q_2$, $Z_1 = X_1R_1$, $Z_2 = X_2R_2$. The variables z, Z_1, Z_2 and w are respectively $(\mathrm{Tn} \times 1)$, $(\mathrm{Tn} \times \mathrm{k}_1)$, $(\mathrm{Tn} \times \mathrm{k}_2)$ and $(\mathrm{Tn} \times 1)$.

Replacing the expression of *w* from (16) in (11), we obtain the concentrated loglikelihood:

$$L_c(\Pi_1, \Pi_2, \Omega) = -\frac{Tn}{2} ln(2\pi) + \frac{T}{2} ln|\Omega^{-1}| - \frac{1}{2}(z - Z_1\Pi_1 - Z_2\Pi_2)'(I_T \otimes \Omega)^{-1}(z - Z_1\Pi_1 - Z_2\Pi_2) \quad (17)$$

Noting that:

$$\begin{aligned} d_{\Pi_1} L_c &= -\frac{1}{2} tr\{2(z - Z_1\Pi_1 - Z_2\Pi_2)'(I_T \otimes \Omega)^{-1}(-Z_1 d\Pi_1)\} \\ &= tr\{(z - Z_1\Pi_1 - Z_2\Pi_2)'(I_T \otimes \Omega)^{-1}Z_1 d\Pi_1\} \end{aligned}$$

$$d_{\Pi_2} L_c = tr\{(z - Z_1\Pi_1 - Z_2\Pi_2)'(I_T \otimes \Omega)^{-1}Z_2 d\Pi_2\}$$

the first order conditions for maximum are:

$$\frac{\partial L_c}{\partial \Pi_1} = Z_1'(I_T \otimes \Omega)^{-1}(z - Z_1\Pi_1 - Z_2\Pi_2) = 0$$
$$\frac{\partial L_c}{\partial \Pi_2} = Z_2'(I_T \otimes \Omega)^{-1}(z - Z_1\Pi_1 - Z_2\Pi_2) = 0$$

Assuming Ω known for the moment, we can rewrite the system as:

$$A_{11}\Pi_1 + A_{12}\Pi_2 = b_1$$

$$A_{21}\Pi_1 + A_{22}\Pi_2 = b_2$$

with $A_{11} = Z_1'(I_T \otimes \Omega)^{-1}Z_1$

$$A_{12} = Z_1'(I_T \otimes \Omega)^{-1}Z_2$$

$$A_{21} = Z_2'(I_T \otimes \Omega)^{-1}Z_1$$

$$A_{22} = Z_2'(I_T \otimes \Omega)^{-1}Z_2$$

$$b_1 = Z_1'(I_T \otimes \Omega)^{-1}z$$

$$b_2 = Z_2'(I_T \otimes \Omega)^{-1}z$$

Since A_{22} , $(A_{11} - A_{12}A_{22}^{-1}A_{21})$ and $(A_{22} - A_{21}A_{11}^{-1}A_{12})$ are non-singular, we can solve the above system to get:

$$\widehat{\Pi}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(b_1 - A_{12}A_{22}^{-1}b_2) \quad (18)$$

$$\widehat{\Pi}_2 = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}(b_2 - A_{21}A_{11}^{-1}b_1) \tag{19}$$

Estimators for *vec(B)* and *vec(*C) are derived from $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ using (14) and (15):

$$vec(\hat{B}) = q_1 + R_1\widehat{\Pi}_1 \qquad and \qquad vec(\hat{C}) = q_1 + R_1\widehat{\Pi}_2 \tag{20}$$

Now, let us derive the likelihood with respect to Ω:

$$\frac{\partial L_c}{\partial \Omega^{-1}} = \frac{T}{2}\frac{\partial ln|\Omega^{-1}|}{\partial \Omega^{-1}} - \frac{1}{2}\frac{\partial (z - Z_1\Pi_1 - Z_2\Pi_2)'(I_T \otimes \Omega^{-1})(z - Z_1\Pi_1 - Z_2\Pi_2)}{\partial \Omega^{-1}}$$

$$= \frac{T}{2}\Omega - \frac{1}{2}\sum_{t=1}^{T}(z_t - Z_{1t}'\Pi_1 - Z_{2t}'\Pi_2)(z_t - Z_{1t}'\Pi_1 - Z_{2t}'\Pi_2)'$$

Using various matrix derivate results.

Setting it to zero leads to:

$$\widehat{\Omega}(\widehat{\Pi}_1, \widehat{\Pi}_2) = \frac{1}{T}\sum_{t=1}^{T}(z_t - Z_{1t}'\widehat{\Pi}_1 - Z_{2t}'\widehat{\Pi}_2)(z_t - Z_{1t}'\widehat{\Pi}_1 - Z_{2t}'\widehat{\Pi}_2)'$$

or

$$\widehat{\Omega} = \frac{1}{T}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t' = \frac{1}{T}\widehat{W}'\widehat{W} \tag{21}$$

with $\widehat{w}_t = z_t - Z_{1t}'\widehat{\Pi}_1 - Z_{2t}'\widehat{\Pi}_2$ and $\widehat{W}' = [\widehat{w}_1 \ \widehat{w}_2 \ldots \widehat{w}_T]$

Noting that the solution for $\widehat{\Omega}$ depends on $\widehat{\Pi}_1$, $\widehat{\Pi}_2$ and vice versa, in practice, an
iterative method must be adopted for a numerical solution, starting with say
$\Omega_0 = I$.

To sum up, we see that our estimation method consists of two stages: the first one estimating the coefficients of $A(L)$ given B and C and the second stage estimating B, C and Ω (by iteration) using the first stage estimator of $A(L)$.

Johansen (1995)'s method can be described in three stages:

1. Estimation of $A(L)$ given B and C (same as ours)
2. Estimation of C given B and under the constraint $C = \alpha\beta'$ (termed as the I(1) model by *Johansen (1995)).*
3. Estimation of B under the additional constraint $\alpha'_{\perp} B \beta_{\perp} = \phi\eta'$ named the $I(2)$ model.

Thus *Johansen (1995)* successively introduces the cointegration constraints in each stage of the estimation method whereas we take account of them right upfront in the specification of the model. Further, in his model the possibility $CI(2,2)$ does not exist except under additional complex constraints whereas in our model, both $CI(2,2)$ and $CI(2,1)$ can be modelled easily.

4. Limiting Distributions

Theorem 1:

$$\begin{pmatrix} T(\hat{\Pi}_1 - \Pi_1) \\ T^2(\hat{\Pi}_2 - \Pi_2) \end{pmatrix} \xrightarrow{L} [\,J \times \left(\int_0^1 \boldsymbol{B}(r)\boldsymbol{B}(r)'dr \otimes \Omega^{-1} \right) \times J'\,]^{-1}$$

$$\times J \times (I_{2n} \otimes \Omega^{-1}) \times \int_0^1 \boldsymbol{B}(r) \otimes dB_0(r)$$

where $J = \begin{pmatrix} R_1'(F(1) \otimes I_n) & 0 \\ 0 & R_2'(F(1) \otimes I_n) \end{pmatrix}$

$\boldsymbol{B}(r)' = [B(r)' \;\; \overline{B}(r)']$ and B_0 are independent Brownian motions.

Proof: See Annex 2.

Using the same arguments as developed by *Phillips* (*1991*) this asymptotic result can be expressed as a mixed normal distribution

$$\begin{pmatrix} T(\hat{\Pi}_1 - \Pi_1) \\ T^2(\hat{\Pi}_2 - \Pi_2) \end{pmatrix} \xrightarrow{L} \int_{G>0} N(0, [J\,(G \otimes \Omega^{-1})\,J']^{-1})dP(G)$$

where $G = \int_0^1 \boldsymbol{B}(r)\boldsymbol{B}(r)'\,dr$ and P is a probability measure associated with G.

For a given G we have $P(G) = 1$ and the asymptotic variance is written as

$$AsyVar(\hat{\Pi}_1, \hat{\Pi}_2) = \begin{pmatrix} T & 0 \\ 0 & T^2 \end{pmatrix}^{-1} [J(G \otimes \Omega^{-1})J'\,]^{-1} \begin{pmatrix} T & 0 \\ 0 & T^2 \end{pmatrix}^{-1}$$

This can be estimated by

$$A\widehat{syV}ar\left(\widehat{\Pi}_1,\widehat{\Pi}_2\right)=\begin{pmatrix}T & 0\\ 0 & T^2\end{pmatrix}^{-1}\left[J\left(\hat{G}\otimes\Omega^{-1}\right)J'\right]^{-1}\begin{pmatrix}T & 0\\ 0 & T^2\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}\tilde{Z}_1'\tilde{Z}_1 & \tilde{Z}_1'\tilde{Z}_2\\ \tilde{Z}_2'\tilde{Z}_1 & \tilde{Z}_2'\tilde{Z}_2\end{pmatrix}$$

Thus

$$A\widehat{syV}ar\left(\widehat{\Pi}_1\right)=\left(\tilde{Z}_1'\widetilde{M}_2\tilde{Z}_1\right)^{-1}\qquad(22)$$

$$A\widehat{syV}ar\left(\widehat{\Pi}_2\right)=\left(\tilde{Z}_2'\widetilde{M}_1\tilde{Z}_2\right)^{-1}\qquad(23)$$

5. Asymptotic comparison between least squares and maximum likelihood estimators

In this section we compare the asymptotic convergences of ordinary least squares and maximum likelihood estimators from the error correction model. Then we demonstrate the asymptotic equivalence between the two methods using the integrated higher order variables. In order to arrive at this result, we first need to obtain the asymptotic convergence of the least squares estimator of *Stock* and *Watson (1993).*

As OLS or GLS is applied block by block by *Stock and Watson (1993),* we will consider the estimation of the last block in which the parameters of the other equations are already substituted by their estimates. More specifically, in our case, it amounts to estimating the third set of equations knowing θ_1.

After a few transformations (see Annex 3), the estimating equation can be writ ten as:

$$y_{3t} = \Theta_0 z_{0t} + \Theta_1 z_{1t} + \Theta_2 z_{2t} + w_{3t} = \Theta z_t + w_{3t} \qquad (24\}$$

Where

1. $\Theta_0 = [A_{30} \quad A_{31} \dots \quad A_{3q}]$ with q a finite positive number.
   $z'_{0t} = [\triangle^2 y'_{1,t-1} \ \triangle^2 y'_{1,t-2}, \dots, \triangle^2 y'_{1,t-1+q}]$ and $z_{0t} \sim I(0)$
2. $\Theta_1 = \delta$, $z'_{1t} = \triangle y'_{1,t-1}$ and $z_{1t} \sim I(1)$
3. $\Theta_2 = [\theta_3 \ \theta_2]$; $z'_{2t} = \left(y'_{1,t-2} \quad y'_{2,t-2}\right)$ and $z_{2t} \sim I(1)$

Let us write $\Theta = [\Theta_0 \quad \Theta_1 \quad \Theta_2]$ and $z'_t = (z'_{0t} \ z'_{1t} \ z'_{2t})$. We have y_{3t} , z_{it} and w_{3t}
$(n_3 \times 1)$, $(k_1 \times 1)$ and $(n_3 \times 1)$ respectively and hence Θ is $(n_3 \times k)$ and $z_1(k \times 1)$
where $k = \sum_{i=0}^{2} k_i$. Writing $\Theta z_t = vec(\Theta z_t) = \left(z'_t \otimes I_{n_3}\right) vec(\Theta)$ we have

$$y_{3t} = \left(z'_t \otimes I_{n_3}\right) vec(\Theta) + w_{3t}$$

with $vec(\Theta') = [vec(\Theta'_0) \ \ vec(\Theta'_1) \ \ vec(\Theta'_2)]$.

Piling together the T observations, the model becomes

$$vec(Y_3') = \left(Z \otimes I_{n_3}\right) vec(\Theta) + vec(W_3') \qquad (25)$$

with $Y_3' = (y_{31} \;\; y_{32} \;....\; y_{3T})$, $Z' = (z_1 \;\; z_2 \;...\; z_T)$, $W_3' = (w_{31} \; w_{32} \;...\; w_{3T})$.

At this stage, we note that given our assumption 1, it can be shown that OLS and GLS are equivalent[2]. Hence, we will only compare OLS and ML estimators here.

Applying ordinary least squares,

$$vec\left(\widehat{\Theta}_{ols}\right) = [\sum_{t=1}^{T} z_t z_t' \otimes I_{n_3}]^{-1} \sum_{t=1}^{T} \left(z_t \otimes I_{n_3}\right) y_{3t} \qquad (26)$$

or substituting for y_{3t}, we have:

$$vec\left(\widehat{\Theta}_{ols}\right) - vec(\Theta) = [\sum_{t=1}^{T} z_t z_t' \otimes I_{n_3}]^{-1} \sum_{t=1}^{T} z_t \otimes w_{3t} \qquad (27)$$

The normalizing factors for these coefficients is given by the matrix

$$\Gamma_T = \begin{pmatrix} T^{\frac{1}{2}} I_{k_0} & 0 & 0 \\ 0 & T I_{k_1} & 0 \\ 0 & 0 & T^2 I_{k_2} \end{pmatrix}$$

Remultiplying by $\Gamma_T \otimes I_{n_1}$, equation (27) becomes

$$\left(\Gamma_T \otimes I_{n_3}\right)[vec(\Theta_{ols}) - vec(\Theta)] = [Q_T \otimes I_{n_3}]^{-1} \varphi_T$$

where $Q_T = \Gamma_T^{-1} (\sum_{t=1}^{T} z_t z_t') \, \Gamma_T^{-1}$ and $\varphi_T = \sum_{t=1}^{T} \Gamma_{-1}^{T} z_t \otimes v_{3t}$

---

[2] Even if we assume serially dependent w_t , it can be shown that OLS and GLS estimators of coefficients of non-stationnary variables are asymptotically equivalent. This result is an extension of *Phillips and Park (1988)* in the higher order case.

Theorem 2:

$$\begin{pmatrix} T\left(vec(\Theta_{1,ols}) - vec(\Theta_1)\right) \\ T^2\left(vec(\Theta_{2,ols}) - vec(\Theta_2)\right) \end{pmatrix} \xrightarrow{L} [\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}^{-1} \otimes I_{n_3}] \begin{pmatrix} \varphi_{1*} \\ \varphi_{2*} \end{pmatrix}$$

Where

1. $Q_{11} \equiv F_1(1) \int_0^1 B(r)B(r)'dr\, F_1(1)'$
2. $Q_{12} = Q'_{21} \equiv F_1(1) \int_0^1 B(r)\overline{B}(r)'dr\, F_*(1)'$
3. $Q_{22} \equiv F_*(1) \int_0^1 \overline{B}(r)\overline{B}(r)'dr\, F_*(1)'$
4. $\varphi_{1*} \equiv [F_1 \otimes I_{n_3}] \int_0^1 B(r) \otimes dB_{03}(r)$
5. $\varphi_{2*} \equiv [F_* \otimes I_{n_3}] \int_0^1 B(r) \otimes dB_{03}(r)$

with

- $F'_* = [F_1(1)' \quad F_2(1)']$
- $B(r)$ is a Brownian motion and $\overline{B}(\mathrm{r}) = \int_0^{\mathrm{r}} \mathrm{B(s)\,ds}$

Proof: See Annex 4.

In order to compare this distribution with that of the maximum likelihood estimators, we need to get the ML estimator of the third equation of the triangular representation (5). This equation can be written as:

$$\triangle^2 y_{3t} = A_{2*}(L) \triangle^2 y_{1,t-1} + \delta \triangle y_{1,t-1} - 2 \triangle y_{3,t-1} + \theta_3 y_{1,t-2} + \theta_2 y_{2,t-2} - y_{3,t-2} + w_{3t}$$

$$= A_*(L)\xi_{t-1} + B_* \triangle y_{t-1} + C_* y_{t-2} + w_{3t} (28)$$

where

1. $A_*(L) = [A_{3*}(L) \;\; 0 \;\; 0]$
2. $B_* = [\delta \;\; 0 \; -2I_{n_3}]$

3. $C_* = [\theta_3 \quad \theta_2 \ - I_{n_3}]$

- $vec(B_*) = \begin{pmatrix} 0 \\ 0 \\ -2vec(I_{n_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2vec(I_{n_3}) \end{pmatrix} + \begin{pmatrix} I_{n_1 n_2} \\ 0 \\ 0 \end{pmatrix} vec(\delta)$

$$= q_{1*} + R_{1*}\Pi_{1*}$$

with

$q_{1*} = \begin{pmatrix} 0 \\ 0 \\ -2vec(I_{n_3} \end{pmatrix}$; $R_{1*} = \begin{pmatrix} I_{n_1 n_3} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{n_1} \\ 0 \\ 0 \end{pmatrix} \otimes I_{n_3}$; $\Pi_{1*} = vec(\delta)$

- $vec(C_*) = \begin{pmatrix} vec(\theta_3) \\ vec(\theta_2) \\ -vec(I_{n_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -vec(I_{n_3}) \end{pmatrix} + \begin{pmatrix} I_{n_1 n_3} & 0 \\ 0 & I_{n_2 n_3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} vec(\theta_3) \\ vec(\theta_2) \end{pmatrix}$

$$= q_{2*} + R_{2*}\Pi_{2*}$$

with

$q_{2*} = \begin{pmatrix} 0 \\ 0 \\ -vec(I_{n_3} \end{pmatrix}$; $R_{2*} = \begin{pmatrix} I_{n_1 n_3} & 0 \\ 0 & I_{n_2 n_3} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{pmatrix} \otimes I_{n_3}$;

$\Pi_{2*} = \begin{pmatrix} vec(\theta_3) \\ vec(\theta_2) \end{pmatrix}$

Applying ML to (28), we obtain

$$\widehat{\Pi}_{1*} = \left(\tilde{Z}'_{1*}\widetilde{M}_{2*}\tilde{Z}_{1*}\right)^{-1}\tilde{Z}'_{1*}\widetilde{M}_{2*}\tilde{z}_{*}$$

$$\widehat{\Pi}_{2*} = \left(\tilde{Z}'_{2*}\widetilde{M}_{1*}\tilde{Z}_{2*}\right)^{-1}\tilde{Z}'_{2*}\widetilde{M}_{1*}\tilde{z}_{*}$$

where $Z_{1*} = \left(M_{\xi} \triangle Y_{-1} \otimes I_{n_3}\right)R_{1*}$, $Z_{2*} = \left(M_{\xi} \triangle Y_{-2} \otimes I_{n_3}\right)R_{2*}$

and $z_{*} = \left(M_{\xi} \otimes I_{n_3}\right)vec(\triangle^2 Y'_3) - \left(M_{\xi} \triangle Y_{-1} \otimes I_{n_3}\right)q_{1*} +$
$\left(M_{\xi} \triangle Y_{-2} \otimes I_{n_3}\right)q_{2*}$

with $\widetilde{M}_{1*} = I_{Tn_3} - \tilde{Z}_{1*}\left(\tilde{Z}'_{1*}\tilde{Z}_{1*}\right)^{-1}\tilde{Z}'_{1*}$

and $\widetilde{M}_{2*} = I_{Tn_3} - \tilde{Z}_{2*}\left(\tilde{Z}'_{2*}\tilde{Z}_{2*}\right)^{-1}\tilde{Z}'_{2*}$

Note That $\widehat{\Pi}_{1*} = vec\left(\widehat{\delta}_{ml}\right)$ and $\widehat{\Pi}_{2*} = \begin{pmatrix} vec\left(\widehat{\theta}_{3,ml}\right) \\ vec\left(\widehat{\theta}_{2,ml}\right) \end{pmatrix}$

Applying the result of Annex 2 (A2.1, A2.2, A2.3 in particular) to the ML estimator, we can obtain their asymptotic convergence as follows:

1. $T\left(\widehat{\Pi}_{1*} - \Pi_{1*}\right) \xrightarrow{L} \Psi_{1*}^{-1}\rho_{1*}$
2. $T^{-2}\left(\widehat{\Pi}_{2*} - \Pi_{2*}\right) \xrightarrow{L} \Psi_{2*}^{-1}\rho_{2*}$

where

- $\Psi_{1*} = R'_{1*}\Psi_{11*}R_{1*} - R'_{1*}\Psi_{12*}R_{2*}[R'_{2*}\Psi_{22*}R_{2*}]^{-1}R'_{2*}\Psi_{21*}R_{1*}$
- $\rho_{1*} = R'_{1*}\rho_{11*} - R'_{1*}\Psi_{12*}R_{2*}[R'_{2*}\Psi_{22*}R_{2*}]^{-1}R'_{2*}\rho_{22*}$
- $\Psi_{2*} = R'_{2*}\Psi_{22*}R_{2*} - R'_{2*}\Psi_{21*}R_{1*}[R'_{1*}\Psi_{11*}R_{1*}]^{-1}R'_{1*}\Psi_{12*}R_{2*}$
- $\rho_{2*} = R'_{2*}\rho_{22*} - R'_{2*}\Psi_{21*}R_{1*}[R'_{1*}\Psi_{11*}R_{1*}]^{-1}R^{1}_{2*}\rho_{11*}$

1. $\Psi_{11*} = F(1)\int_0^1 B(r)B(r)'dr\, F(1)' \otimes \Omega_3^{-1}$

$$\Rightarrow R'_{1*}\Psi_{11*}R_{1*} = F_1(1)\int_0^1 B(r)B(r)'dr\, F_1(1)' \otimes \Omega_3^{-1}$$
$$= Q_{11} \otimes \Omega_3^{-1}$$

2. $\Psi_{12*} = F(1)\int_0^1 B(r)\overline{B}(r)'dr\, F(1)' \otimes \Omega_3^{-1}$

$$\Longrightarrow R_{1*}'\Psi_{12*}R_{2*} = F_1(1)\int_0^1 B(r)\bar{B}(r)'dr\, F_*(1)' \otimes \Omega_3^{-1}$$
$$= Q_{12} \otimes \Omega_3^{-1}$$

3. $\Psi_{22*} = F(1)\int_0^1 \overline{B}(r)\overline{B}(r)'dr\, F(1)' \otimes \Omega_3^{-1}$

$$\Longrightarrow R_{2*}'\Psi_{22*}R_{2*} = F_*(1)\int_0^1 \bar{B}(r)\bar{B}(r)'dr\, F_*(1)' \otimes \Omega_3^{-1}$$
$$= Q_{22} \otimes \Omega_3^{-1}$$

4. $\Psi_{21*} = F(1)\int_0^1 \overline{B}(r)B(r)'dr\, F(1)' \otimes \Omega_3^{-1}$

$$\Longrightarrow R_{2*}'\Psi_{21*}R_{1*} = F_1(1)\int_0^1 \bar{B}(r)B(r)'dr\, F_*(1)' \otimes \Omega_3^{-1}$$
$$= Q_{21} \otimes \Omega_3^{-1}$$

5. $\rho_{11*} = [F(1) \otimes \Omega_3^{-1}]\int_0^1 B(r) \otimes dB_{03}(r)'$

$$\Longrightarrow R_{1*}'\rho_{11*} = [F_1(1) \otimes \Omega_3^{-1}]\int_0^1 B(r) \otimes dB_{03}(r)$$

Hence

$R_{1*}'\rho_{11*} = \left(I_{n_1} \otimes \Omega_3^{-1}\right)\left[F_1(1) \otimes I_{n_3}\right]\int_0^1 B(r) \otimes dB_{03}(r) = (I_{n_1} \otimes \Omega_3^{-1})\,\varphi_{1*}$

6. $\rho_{22*} = [F(1) \otimes \Omega_3^{-1}]\int_0^1 \overline{B}(r) \otimes dB_{03}(r)$

$$\Longrightarrow R_{2*}'\rho_{22*} = [F_*(1) \otimes \Omega_3^{-1}]\int_0^1 \bar{B}(r) \otimes dB_{03}(r)$$
$$= \left(I_{n_1+n_2} \otimes \Omega_3^{-1}\right)\left[F_*(1) \otimes I_{n_3}\right]\int_0^1 \overline{B}(r) \otimes dB_{03}(r)$$
$$= \left(I_{n_1+n_2} \otimes \Omega_3^{-1}\right)\varphi_{2*}$$

(a) $\Psi_{1*} = Q_{11} \otimes \Omega_3^{-1} - (Q_{12} \otimes \Omega_3^{-1})(Q_{22} \otimes \Omega_3^{-1})^{-1}(Q_{21} \otimes \Omega_3^{-1})$
$\quad = (Q_{11} - Q_{12}\, Q_{22}^{-1}Q_{21}) \otimes \Omega_3^{-1}$

(b) $\rho_{1*} = \left(I_{n_1} \otimes \Omega_3^{-1}\right)\varphi_{1*} - (Q_{12} \otimes \Omega_3^{-1})(Q_{22} \otimes \Omega_3^{-1})^{-1}\left(I_{n_1+n_2} \otimes \Omega_3^{-1}\right)\varphi_{2*}$
$\quad = \left(I_{n_1} \otimes \Omega_3^{-1}\right)\varphi_1 - (Q_{12}Q_{22}^{-1} \otimes \Omega_3^{-1})\varphi_{2*}$

(c) $\Psi_{1*}^{-1}\rho_{1*} = [(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1} \otimes \Omega_3]\left[\left(I_{n_1} \otimes \Omega_3^{-1}\right)\varphi_{1*} - (Q_{12}Q_{22}^{-1} \otimes \Omega_3^{-1})\varphi_{2*}\right]$

$$= \left[(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1} \otimes I_{n_3}\right]\varphi_{1*} - \left[(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1}\mathrm{Q}_{12}\mathrm{Q}_{22}^{-1} \otimes I_{n_3}\right]\varphi_{2*}$$

$$\Rightarrow T\left(\widehat{\Pi}_{1*} - \Pi_{1*}\right) \overset{L}{\rightarrow} \left[(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1} \otimes I_{n_3}\right]\left[\varphi_{1*} - \left(\mathrm{Q}_{12}\mathrm{Q}_{22}^{-1} \otimes I_{n_3}\right)\varphi_{2*}\right]$$

Therefore, comparing the above limit with (A4.3) in Annex 4 , we can conclude that

$$T\left(\widehat{\Pi}_{1*} - \Pi_{1*}\right) \simeq^{as} T\left(vec\left(\widehat{\Theta}_{1,ols}\right) - vec(\Theta_1)\right) \tag{29}$$

(d) $\Psi_{2*} = Q_{22} \otimes \Omega_3^{-1} - (Q_{21} \otimes \Omega_3^{-1})(Q_{11} \otimes \Omega_3^{-1})^{-1}(Q_{12} \otimes \Omega_3^{-1})$
$= (Q_{22} - Q_{21}\ Q_{11}^{-1}Q_{12}) \otimes \Omega_3^{-1}$

(e) $\rho_{2*} = \left(I_{n_1+n_2} \otimes \Omega_3^{-1}\right)\varphi_{2*} - (Q_{21} \otimes \Omega_3^{-1})(Q_{11} \otimes \Omega_3^{-1})^{-1}\left(I_{n_1} \otimes \Omega_3^{-1}\right)\varphi_{1*}$
$= \left(I_{n_1+n_2} \otimes \Omega_3^{-1}\right)\varphi_{2*} - (Q_{21}Q_{11}^{-1} \otimes \Omega_3^{-1})\varphi_{1*}$

(f) $\Psi_{2*}^{-1}\rho_{2*} = [(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})^{-1} \otimes \Omega_3]\left[\left(I_{n_1+n_2} \otimes \Omega_3^{-1}\right)\varphi_{2*} - (Q_{21}Q_{11}^{-1} \otimes \Omega_3^{-1})\varphi_{1*}\right]$

$$= \left[(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})^{-1} \otimes I_{n_3}\right]\varphi_{2*} - \left[(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})^{-1}\mathrm{Q}_{21}\mathrm{Q}_{11}^{-1} \otimes I_{n_3}\right]\varphi_{1*}$$

$$\Rightarrow T^2\left(\widehat{\Pi}_{2*} - \Pi_{2*}\right) \overset{L}{\rightarrow} \left[(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})^{-1} \otimes I_{n_3}\right]\left[\varphi_{2*} - \left(\mathrm{Q}_{21}\mathrm{Q}_{11}^{-1} \otimes I_{n_3}\right)\varphi_{1*}\right]$$

Once again comparing the above with $(A4.4)$ in Annex 4 , we can write :

$$T^2\left(\widehat{\Pi}_{2*} - \Pi_{2*}\right) \simeq^{as} T^2\left(vec\left(\widehat{\Theta}_{2,ols}\right) - vec(\Theta_2)\right) \tag{30}$$

Recalling $\Theta_1 = \delta$; $\Theta_2 = [\theta_3 \;\; \theta_2]$; $\Pi_{1*} = vec(\delta_{ml})$;

$$\Pi_{2*} = \begin{pmatrix} vec(\theta_{3,ml}) \\ vec(\theta_{2,ml}) \end{pmatrix}$$

It results from (29) and (30) :

- $T\left(vec(\hat{\delta}_{ml}) - vec(\delta)\right) \simeq^{as} T\left(vec(\hat{\delta}_{ols}) - vec(\delta)\right)$
- $T^2\left(vec(\hat{\theta}_{3,ml}) - vec(\theta_3)\right) \simeq^{as} T^2\left(vec(\hat{\theta}_{3,ols}) - vec(\theta_3)\right)$
- $T^2\left(vec(\hat{\theta}_{2,ml}) - vec(\theta_2)\right) \simeq^{as} T^2\left(vec(\hat{\theta}_{2,ols}) - vec(\theta_2)\right)$

6. Conclusion

we would like to point out that our approach of deriving an error correction model starting from the triangular representation of *Stock* and *Watson* (*1993*) for higher order (co)integrated systems, already incorporates the cointegration restrictions that need to be separately imposed in *Johansen 's* (*1995*) specification. We apply the maximum likelihood method to the model obtained in this way and compare the asymptotic properties of the resulting estimators with those of the least squares estimators of *Stock* and *Watson* (*1993*) proving their equivalence as far as the coefficients of non-stationary variables are concerned. It can be added that our method can be easily implemented for any practical application as explicit solutions are given. Moreover, extension to cointegrated systems of any higher order can be achieved without much difficulty.

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Annex 1 : Derivation of the error correction model

We start from the triangular representation (6) that we rewrite below for easy reference:

$$\begin{cases} \Delta^2 y_{1t} = v_{1t} \\ \Delta y_{2t} = D_1(L)\Delta^2 y_{1t} + \theta_1 + \Delta y_{1t} + v_{2t} \\ y_{3t} = D_2(L)\Delta^2 y_{1t} + D_3(L)(\Delta y_{2t} - \theta_1 \Delta y_{1t}) + \theta_2 y_{2t} + \theta_3 y_{1t} \\ \quad + \theta_4 \Delta y_{1t} + v_{3t} \end{cases}$$

The first equation stays as it is :

$$\triangle^2 y_{1t} = v_{1t}$$

Let us rewrite the second in the following way:

$$\begin{aligned} \Delta^2 y_{2t} = & \quad D_1(L)\Delta^3 y_{1t} + \theta_1 \Delta^2 y_{1t} + \Delta v_{2t} \\ = & \quad D_1(L)\left(\Delta^2 y_{1t} - \Delta^2 y_{1,t-1}\right) + \theta_1 v_{1t} + \left(v_{2t} + v_{2,t-1}\right) \\ = & \quad D_1(L)\left(\Delta^2 y_{1t} - \Delta^2 - y_{1,t-1}\right) + \theta_1 v_{1t} + v_{2t} - \left(\Delta y_{2,t-1} - D_1(L)\Delta^2 y_{2,t-1} - \theta_1 \Delta y_{1,t-1}\right) \\ = & \; D_1(L)v_{1t} + \theta_1 v_{1t} + v_{2t} - \left(\Delta y_{2,t-1} - \theta_1 \Delta y_{2,t-1} - \theta_1 \Delta y_{1,t-1}\right) \end{aligned}$$

$$= d_{10} v_{1y} + D_{11}(L)v_{1,t-1} + \theta_1 v_{1t} + v_{2t} - \left(\Delta y_{2,t-1} - \theta_1 \Delta y_{1,t-1}\right)$$

$$\Longrightarrow \Delta^2 y_{2t} = D_{11}(L)\Delta^2 y_{1,t-1} - \left(\Delta y_{2,t-1} - \theta_1 \Delta y_{1,t-1}\right) + (d_{10} + \theta_1) v_{1t} + v_{2t}$$

Where $D_1(L) = d_{10} + LD_{11}(L)$

Now, let us consider the third equation. We take its second difference:

$$D^2 y_{3t} = D_2(L)\Delta^4 y_{1t} + D_3(L)(\Delta^3 y_{2t} - \theta_1 \Delta^3 y_{1t}) + \theta_2 \Delta^2 y_{2t} + \theta_3 \Delta^2 y_{1t} + \theta_4 \Delta^3 y_{1t} + \Delta^2 v_{3t}$$

1. $D_2(L)\Delta^4 y_{1t} = D_2(L)\Delta^2 y_{1t} - 2 D_2(L)\Delta^3 y_{1,t-1} - D_2(L)\Delta^2 y_{1,t-2}$
2. $D_3(L)(\Delta^3 y_{2t} - \theta_{1t}) = D_3(L)(\Delta y_{2t} - \theta_1 \Delta y_{1t}) - 2D_3(L)\left(\Delta y_{2,t-1} \theta_1 \Delta y_{1,t-1}\right) - D_3(L)\left(\Delta y_{2,t-2} - \theta_1 \Delta y_{1,t-2}\right)$

3. $\theta_2\,\Delta^2 y_{2t} = \theta_2\big[D_{11}(L)\Delta^2 y_{1,t-1} - \big(\Delta\, y_{2,t-1} - \theta_1\,\Delta\, y_{1,t-1}\big) + (\,d_{10} + \theta_1)v_{1t}\, v_{2t}\big]$
4. $\theta_3\,\Delta^2\, y_{1t} = \theta_3\, v_{1t}$
5. $\theta_4\,\Delta^3\, y_{1t} = \theta_4\left(\Delta^2\, y_{1t} - \Delta^2\, y_{1,t-1}\right) = \theta_4\, v_{1t} - \theta_4\,\Delta^2\, y_{1,t-1}$
6. $\Delta^2\, v_{3t} = v_{3t} - 2\,\Delta\, v_{3,t-1} - v_{3,t-2}$

$$= v_{3t} - 2\big[\Delta\, y_{3,t-1} - D_2\,(L)\Delta^3\, y_{1,t-1} - D_3\,(L)\big(\,\Delta^2\, y_{2,t-1} - \theta_1\,\Delta^2\, y_{1,t-1}\,\big) - \theta_2\,\Delta\, y_{2,t-1} - \theta_3\,\Delta\, y_{1,t-1} - \theta_4\,\Delta^2\, y_{1,t-1}\big] - \big[y_{3,t-2} - D_2\,(L)\Delta^2\, y_{1,t-2} - D_3\,(L)\big(\Delta\, y_{2,t-2} - \theta_1\,\Delta\, y_{1,t-2}\big) - \theta_2\, y_{2,t-2} - \theta_3\, y_{1,t-2} - \theta_4\,\Delta y_{1,t-2}\big]$$

$$\Rightarrow \Delta^2\, y_{3t} = D_2\,(\,L)\Delta^2\, y_{1t} + D_3\,(L)(\Delta y_{2t} - \theta_1\, y_{1t})$$

$$-\,\theta_2\,\left(\Delta\, y_{2,t-1} - \theta_1\,\Delta\, y_{1,t-1}\right) + \theta_2\,\theta_1\, v_{1t} + \theta_2\, v_{2t}$$

$$+\,\theta_3 v_{1t} + \theta_4\, v_{1t} - \theta_4\,\Delta^2\, y_{1,t-1}$$

$$+v_{3t} - 2\big(\Delta y_{3,t-1} - \theta_2\,\Delta y_{2,t-1} - \theta_3\,\Delta y_{1,t-1} - \theta_3\,\Delta y_{1,t-1} - \theta_4\,\Delta^2\, y_{2,t-1} - \theta_4\,\Delta^2\, y_{1,t-1}\big)$$

$$-\big(y_{3,t-2} - \theta_2\, y_{2,t-2} - \theta_3\, y_{1,t-2} - \theta_4\,\Delta y_{1,t-2}\big)$$

7. Note that $D_2\,(L)\,D^2\, y_{1t} = d_{20}\,\Delta^2\, y_{1t} + D_{21}\,(\,L)\Delta^2\, y_{1,t-1} = d_{20}\, v_{1t} + D_{21}\,(L)\,\Delta^2\, y_{1,t-1}$

with $D_2\,(L) = d_{20} + LD_{21}\,(L)$

8. $D_3\,(L)(\Delta\, y_{2t} - \theta_1\,\Delta\, y_{1t}) = D_3\,(L)\,\Delta\, y_{2t} - D_3\,(L)\,\theta_1\,\Delta\, y_{1t}$

$$= D_3\,(1)\Delta\, y_{2,t-1} + D_3^{**}\,(L)\,\Delta^2\, y_{2t} - D_3\,(1)\,\theta_1\,\Delta\, y_{1,t-1} - D_3^{**}\,(L)\theta_1\,\Delta^2\, y_{1t}$$

Using $D_3\,(L) = LD_3\,(1) + (1) + (1-L)D_3^{**}(L)$

The decomposition $D_3^{**}\,(L) = D_{30}^{**}\,(L)$ *enables us to write* :

$$D_3^{**}\,(L)\,\Delta^2\, y_{2t} = d_{30}^{**}\,\Delta^2\, y_{2t} + D_{31}^{**}\,(L)\,\Delta^2\, y_{2,t-1}$$

$$= d_{30}^{**}\,\big[D_{11}\,(L)\Delta^2\, y_{1,t-1} + \theta_1\,\Delta y_{2,t-1} + (d_{10} + \theta_1)v_{1t} + v_{2t}\,\big]$$

$$+\, D_{31}^{**}\,(L)\,\Delta^2\, y_{2,t-1}$$

$$= d_{30}^{**}\ D_{11}\ (L)\Delta^2 y_{1,t-1} + D_{31}^{**}\ (L)\Delta^2\ y_{2,t-1} + d_{30}^{**}\ \theta_1\ \Delta y_{1,t-1} - d_{30}^{**}\ \Delta y_{2,t-1} + d_{30}^{**}\ (d_{10} + \theta_1)v_{1t} + d_{30}^{**}\ v_{2t}$$

Using $D_3^{**}(L)\,\theta_1\,\Delta^2 y_{1t} = d_{30}^{**}\ \theta_1\, v_{1t} + D_{31}^{**}\ (L)\theta_1\ \Delta^2\ y_{1,t-1}$

Thus

$$D_3\ (L)(\Delta y_{2t} - \theta_1 \Delta y_{1t})\ [d_{30}^{**}\ D_{11}(L) - D_{31}^{**}\ (L)\ \theta_1]\Delta^2 y_{1,,t-1} + D_{31}^{**}(L)\,\Delta^2\ y_{2,t-1} + [d_{30}^{**}\theta_1 - D_3(1)\theta_1]\Delta y_{1,t-1} + [D_3\ (1) - d_{30}^{**}]\Delta y_{2,t-1} + d_{30}^{**}\ d_{10}\, v_{1t} + d_{30}^{**}\, v_{2t}$$

Combining results (1) to (8), we can rewrite $\Delta^2 y_{3t}$ as :

$$\Delta^2 y_{3t} = [D_{21}\ (L) + d_{30}^{**}\ D_{11}\ (L) - D_{31}^{**}(L)\ \theta_2\ D_{11}\ (L)]\Delta^2 y_{1,t-1} + D_{31}^{**}\ (L)\,\Delta^2\ y_{2,t-1}$$

$$+ [d_{30}^{**}\theta_1 - D_3\ (1)\ \theta_1 + \theta_2\,\theta_1 + 2\,\theta_3 + \theta_4]\Delta y_{1,t-1} + [D_3\,(1) - d_{30}^{**} + \theta_2]\Delta y_{2,t-1} - 2\,\Delta y_{3,t-1} + \theta_3\, y_{1,t-2} + \theta_2\, y_{2,t-2} - y_{3,t-2}$$

$$+(d_{20} + d_{30}^{**}\ d_{10} + \theta_2\, d_{10} + \theta_2\,\theta_1 + \theta_3 + \theta_4)v_{1t} + (d_{30}^{**} + \theta_2)v_{2t} + v_{3t}$$

To sum up, the above transformation lead to the following system :

(a) $\Delta^2 y_{2t} = v_{1t} = w_{1t}$
(b) $\Delta^2 y_{2t} = A_1\ (L)\ \Delta^2 y_{1,t-1} + \theta_1\, y_{1,t-1} - \Delta y_{2,t-1} + w_{2t}$
With

$A_1(L) = D_{11}(L)$ and $w_{2t} = (d_{10} + \theta_1)v_{1t} + v_{2t}$

(c) $\Delta^2 y_{3t} = A_2(L)\Delta^2 y_{1,t-1} + A_3\ (L)\Delta^2 y_{2,t-1}$
$+ \delta_1\ \Delta y_{1,t-1}\,\delta_2\,\Delta y_{2,t-1} - 2\,\Delta y_{3,t-1}$

$+\theta_3\ y_{1,t-2} + \theta_2\ y_{2,t-2} - y_{3,t-2} + w_{3t}$

With $A_2(L) = D_{21}(L) + d_{30}^{**} D_{11}(L) - D_{31}^{**}(l)\theta_1 + \theta_2\, D_{11}(L)$

$A_3\ (L) = D_{31}^{**}\ (L)$

$\delta_1 = d_{30}^{**}\theta_1 - D_3(1)\theta_1 + \theta_2\,\theta_1 + 2\,\theta_3 + \theta_4$

$\delta_2 = D_3(1) - d_{30}^{**} + \theta_2$

$$w_{3t} = (D_{20} + d_{30}^{**} d_{10} + \theta_2 d_{10} + \theta_2 \theta_1 + \theta_3 + \theta_4) v_{1t} + (d_{30}^{**} + \theta_2) v_{2t} + v_{3t}$$

Piling together the three equations (a), (b) and (c), we obtain the error correction model corresponding to the triangular representation given in the beginning of this annex :

$$\Delta^2 y_{2t} = A(L)\, \Delta^2 y_{t-1} + J\Theta' x_{t-1} + \omega_t$$

with $A(L) = \begin{pmatrix} 0 & 0 & 0 \\ A_1(L) & 0 & 0 \\ A_2(L) & A_3(L) & 0 \end{pmatrix}$

$$J = \begin{pmatrix} 0 & 0 & 0 \\ I_{n_3} & 0 & 0 \\ 0 & I_{n_3} & I_{n_3} \end{pmatrix}$$

$$\Theta' = \begin{pmatrix} \theta_1 & -I_{n_2} & 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & -2I_{n_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_3 & \theta_2 & -I_{n_3} \end{pmatrix}$$

$$\Delta^2 y_t' = (\Delta^2 y_{1t}', \Delta^2 y_{2t}', \Delta^2 y_{3t}');\ \Delta^2 y_{t-1}' = \left(\Delta^2 y_{1,t-1}', \Delta^2 y_{2,t-1}', \Delta^2 y_{3,t-1}'\right)$$

$$\Delta y_{t-1}' = \left(\Delta y_{1,t-1}', \Delta y_{2,t-1}', \Delta y_{3,t-1}'\right);\quad y_{t-2}' = \left(y_{1,t-2,}', y_{2,t-2}', y_{3,t-2}'\right)$$

$$x_{t-1} = (\Delta y_{t-1}, y_{t-2}) \ \ and\ \ w_t' = (\omega_{1t}', \omega_{2t}', \omega_{3t}')$$

Annex 2 : asymptotic properties of ML estimators

For asymptotic calculations, it will be more suitable to write $\widehat{\prod}_1$ *and* $\widehat{\prod}_2$ in terms of Z_1 *and* Z_2. we have

- $$A_{11} - A_{12}A_{22}^{-1}A_{21} = \tilde{Z}_1'\tilde{Z}_1 - \tilde{Z}_1'\tilde{Z}_2'\left(\tilde{Z}_2'\tilde{Z}_2\right)^{-1}\tilde{Z}_2'\tilde{Z}_1$$
$$= \tilde{Z}_1'\left[I_{Tn} - \tilde{Z}_2\left(\tilde{Z}_2'\tilde{Z}_2\right)^{-1}\tilde{Z}_2'\right]\tilde{Z}_1 = \tilde{Z}_1'\widetilde{M}_2\tilde{Z}_1$$

with $\tilde{Z}_1 = \left(I_T \otimes \Omega^{-\frac{1}{2}}\right)Z_1 = \left(M_\xi \Delta Y_{-1} \otimes \Omega^{-\frac{1}{2}}\right)R_1$

$$\tilde{Z}_2 = \left(I_T \otimes \Omega^{-\frac{1}{2}}\right)Z_2 = \left(M_\xi \Delta Y_2 \otimes \Omega^{-\frac{1}{2}}\right)R_2$$

and $\widetilde{M}_2 = I_{Tn} - \tilde{Z}_2\left(\tilde{Z}_2'\tilde{Z}_2\right)^{-1}\tilde{Z}_2'$.

$\tilde{Z}_1$and $\tilde{Z}_2$ *are respectively* $(Tn \times k_1)$ *and* $(Tn \times k_2)$ *and of full rank* k_1 *and* k_2. $\widetilde{M}_2$ *is idempotent with*

rank $(\widetilde{M}_2) = tr\left[I_{Tn} - \tilde{Z}_2\left(\tilde{Z}_2'\tilde{Z}_2\right)^{-1}Z_2'\right] = tr(I_{Tn}) - tr[(\tilde{z}_2'\tilde{z}_2)(\tilde{z}_2'z_2)^{-1}]$

$$= Tn - tr\left(I_{k_2}\right) = Tn - k_2$$

thus $\tilde{Z}_1'\widetilde{M}_2\tilde{Z}_1$ *is a positive definite matrix* $(k_1 \times k_1)$. therefore, $(A_{11} - A_{12}A_{22}^{-1}A_{21})$ is positive definite and hence non-singular. Further,

- $$b_1 - A_{11}A_{22}^{-1}b_2 = \widetilde{Z}'_1\left(I_T \otimes \Omega^{-\frac{1}{2}}\right)z - \widetilde{Z}'_1\widetilde{Z}_2\left(\widetilde{Z}'_2\widetilde{Z}_2\right)^{-1}\widetilde{Z}'_2\left(I_T \otimes \Omega^{-\frac{1}{2}}\right)z$$
$$= \left[\widetilde{Z}'_1 - \widetilde{Z}'_2\widetilde{Z}_2\left(\widetilde{Z}'_2\widetilde{Z}_2\right)^{-1}Z_2'\right]\left(I_T \otimes \Omega^{-\frac{1}{2}}\right)z$$
$$= \left(\widetilde{Z}'_1 - \widetilde{Z}'_1 + \tilde{Z}_1'\widetilde{M}_2\right)\left(I_T \otimes \Omega^{-\frac{1}{2}}\right)z$$
$$= \widetilde{Z}'_1\widetilde{M}_2\tilde{z}$$

with $\tilde{z} = \left(I_T \otimes \Omega^{-\frac{1}{2}}\right)z$

with these results, $\widehat{\Pi}_1$ *can be rewritten as* :

$$\widehat{\Pi}_1 = \left(\widetilde{Z}'_1 \widetilde{M}_2 \widetilde{Z}_1\right)^{-1} \widetilde{Z}'_1 \widetilde{M}_2 \widetilde{z}$$

Similarly,

$$\widehat{\Pi}_2 = \left(\widetilde{Z}'_2 \widetilde{M}_2 \widetilde{Z}_2\right)^{-1} \widetilde{Z}'_2 \widetilde{M}_1 \widetilde{z}$$

with $\widetilde{M}_1 = I_{Tn} - \tilde{Z}_1\left(\tilde{Z}'_1\tilde{Z}_1\right)^{-1}\tilde{Z}'_1$ an idempotent matrix of rank $Tn - k_1$.

Let us write down $\widehat{\Pi}_1 - \Pi_1$ and $\widehat{\Pi}_2 - \Pi_2$. Premultiplying $w = z - Z_1\Pi_1 - Z_2\Pi_2$ by $\left(I_T \otimes \Omega^{-1/2}\right)$, we have

$$\widetilde{w} = \left(I_T \otimes \Omega^{-\frac{1}{2}}\right) w$$

$$= \left(I_T \otimes \Omega^{-\frac{1}{2}}\right) z - \left(I_T \otimes \Omega^{-\frac{1}{2}}\right) Z_1\Pi_1 - \left(I_T \otimes \Omega^{-\frac{1}{2}}\right) Z_2\Pi_2$$

$$\Longrightarrow \tilde{w} = \tilde{z} - \tilde{Z}_1\Pi_1 - \tilde{Z}_2\Pi_2$$

Substituting this result, we get

$$\widehat{\Pi}_1 = \left(\tilde{Z}'_1\widetilde{M}_2\tilde{Z}_1\right)^{-1}\tilde{Z}'_1\widetilde{M}_2\left(\widetilde{w} + \tilde{Z}_1\Pi_1 + \tilde{Z}_2\Pi_1\right)$$

$$=\left(\tilde{Z}'_1\widetilde{M}_2\tilde{Z}_1\right)^{-1}\tilde{Z}'_1\widetilde{M}_2\widetilde{w} + \Pi_1$$

$$\Longrightarrow \widehat{\Pi}_1 - \Pi_1 = \left(\tilde{Z}'_1\widetilde{M}_2\tilde{Z}_1\right)^{-1}\tilde{Z}'_1\widetilde{M}_2\widetilde{w}$$

Similarly for $\widehat{\Pi}_2$ is T and for $\widehat{\Pi}_2$ we can write

$$\widehat{\Pi}_2 - \Pi_2 = \left(\tilde{Z}'_2\widetilde{M}_1\tilde{Z}_2\right)^{-1}\tilde{Z}'_2\widetilde{M}_1\widetilde{w}$$

The normalizing factor for $\widehat{\Pi}_1$ is T and for $\widehat{\Pi}_2$ is T^2 the former being associated with variables integrated of order 1 and the latter with I(2) variables. Let us denote

$$T\left(\widehat{\Pi}_1 - \Pi_1\right) = \psi_{1T}^{-1}\,\rho_{2T} \qquad , \qquad T^2\left(\widehat{\Pi}_2 - \Pi_2\right) = \psi_{2T}^{-1}\,\rho_{2T}$$

1. $\psi_{1T} = T^{-2}\bar{Z}'_1\bar{M}_2\bar{Z}_1 = T^{-2}\bar{Z}'_1\bar{Z}_1 - T^{-3}\bar{Z}'_1\bar{Z}_2(T^{-4}\bar{Z}'_2\bar{Z}_2)^{-1}T^{-3}\bar{Z}'_2\bar{Z}_1$

$$= R'_1(T^{-2}\Delta Y'_{-1}M_\varepsilon\Delta Y_{-1} \otimes \Omega^{-1})R_1 - R'_1(T^{-3}\Delta Y'_{-1}M_\varepsilon Y_{-2} \otimes \Omega^{-1})R_2$$

$$.\left[R'_2\left(T^{-4}Y'_{-2}M_\xi Y_{-2} \otimes \Omega^{-1}\right)R_2\right]^{-1}R'_2\left(T^{-3}Y'_{-2}M_\xi\Delta Y_{-1} \otimes \Omega^{-1}\right)R_1$$

2. $\psi_{2T} = T^{-4}\bar{Z}'_2\bar{M}_1\bar{Z}_2 = T^{-4}\bar{Z}'_2\bar{Z}_2 - T^{-3}\bar{Z}'_2\bar{Z}_1(T^{-2}\bar{Z}'_1\bar{Z}_1)^{-1}T^{-3}\bar{Z}'_1\bar{Z}_2$

$$= R'_2(T^{-4}Y'_{-2}M_\varepsilon Y_{-2} \otimes \Omega^{-1})R_2 - R'_2(T^{-3}Y'_{-2}M_\varepsilon \Delta Y_{-1} \otimes \Omega^{-1})R_1 . [R'_1(T^{-2}\Delta Y'_{-1}M_\varepsilon \Delta Y_{-1} \otimes \Omega^{-1})R'_1]^{-1}R'_1(T^{-3}\Delta Y'_{-1}M_\varepsilon Y_{-2} \otimes \Omega^{-1})R_2$$

3. $\rho_{1T} = T^{-1}\bar{Z}'_1\bar{M}_2\bar{w} = R'_1(T^{-1}\Delta Y'_{-1}M_\varepsilon \otimes \Omega^{-1})w - R'_1(T^{-3}\Delta Y'_{-1}M_\varepsilon Y_{-2} \otimes \Omega^{-1})R_2 . [R'_2(T^{-4}Y'_{-2}M_\varepsilon Y_{-2} \otimes \Omega^{-1})R_2]^{-1}R'_2(T^{-2}Y'_{-2}M_\varepsilon \otimes \Omega^{-1})w$

4. $\rho_{2T} = T^{-2}\bar{Z}'_2\bar{M}_1\bar{w} = R'_2(T^{-2}Y'_{-2}M_\varepsilon \otimes \Omega^{-1})w - R'_2(T^{-3}Y'_{-2}M_\varepsilon \Delta Y'_{-1} \otimes \Omega^{-1})R_1 . [R'_1(T^{-2}\Delta Y'_{-1}M_\varepsilon \Delta Y_{-1} \otimes \Omega^{-1})R_1]^{-1}R'_1(T^{-1}\Delta Y'_{-1}M_\varepsilon \otimes \Omega^{-1})w$

Defining ψ_{11T}, ψ_{12T}, ψ_{21T}, ψ_{22T}, ρ_{11T}, ρ_{22T} as follows

(a) $\psi_{11T} = T^{-2}\Delta Y'_{-1}M_\xi \Delta Y_{-1} \otimes \Omega^{-1}$

(b) $\psi_{12T} = T^{-3}\Delta Y'_{-1}M_\xi Y_{-2} \otimes \Omega^{-1}$

(c) $\psi_{21T} = T^{-3}Y'_{-2}M_\xi \Delta Y_{-1} \otimes \Omega^{-1}$

(d) $\psi_{22T} = T^{-4}Y'_{-2}M_\xi Y_{-2} \otimes \Omega^{-1}$

(e) $\rho_{11T} = \left(T^{-1}\Delta Y'_{-1}M_\xi \otimes \Omega^{-1}\right)w$

(f) $\rho_{22T} = \left(T^{-2}Y'_{-2}M_\xi \otimes \Omega^{-1}\right)w$

We obtain

1. $\psi_{1T} = R'_1\psi_{11T}R_1 - R'_1\psi_{12T}R_2[R'_2\psi_{22T}R_2]^{-1}R'_2\psi_{21T}R_1$
2. $\psi_{2T} = R'_2\psi_{22T}R_2 - R'_2\psi_{21T}R_1[R'_1\psi_{11T}R_1]^{-1}R'_1\psi_{12T}R_2$
3. $\rho_{1T} = R'_1\rho_{11T} - R'_1\Psi_{12T}R_2[R'_2\Psi_{22T}R_2]^{-1}R'_2\,\rho_{22T}$
4. $\rho_{2T} = R'_2\rho_{22T} - R'_2\psi_{21T}R_1[R'_1\psi_{11T}R_1]^{-1}R'_1\rho_{11T}$

Let us now go on to calculate their limits.

Let $S_{1t} = \sum_{s=1}^{t} \varepsilon_s$ and $S_{2t} = \sum_{s=1}^{t} S_{1s}$ with $\varepsilon_t = S_{1t} = S_{2t} = 0$ for t < 0.

Knowing that

$\Delta^2 y_t = F(L)\varepsilon_t$, we have $\Delta^2 y_{t-i} = F(L)\varepsilon_{t-i} \ \forall i = 0, 1, 2 \ldots$

With $\varepsilon_{t-i} = 0$ if $t - i \leq 0$ and $F(1) = \sum_{j=0}^{\infty} f_j < \infty$

And we can write

$$\Delta_{t-i} = \Delta y_{t-i-1} + F(L)\varepsilon_{t-i}$$

By recurrence, we obtain

$$\Delta y_{t-i} = \sum_{s=1}^{t-i} F(L)\varepsilon_s$$

In this equation, let us replace $F(L)$ by $F(1) + (1 - L)F^*(L)$

$$\Delta y_{t-i} = F(1)\sum_{s=1}^{t-i} \varepsilon_s + \sum_{s=1}^{t-i} F^*(L)\Delta\varepsilon_s$$

Since

$$\sum_{s=1}^{t-i} F^*(L)\Delta\varepsilon_s = \sum_{s=1}^{t-i}.\left(\sum_{j=0}^{\infty} f_j^* \Delta\varepsilon_{s-j}\right) = \sum_{j=0}^{\infty} f_j^* \left(\sum_{s=1}^{t-i} \Delta\varepsilon_{s-j}\right)$$

$$= \sum_{j=0}^{\infty} f_j^* \varepsilon_{t-i-j} = F^*(L)\varepsilon_{t-i}$$

Therefore

$$\Delta y_{t-i} = F(1)S_{1,t-i} + F^*(L)\varepsilon_{t-i}$$

Expanding Δy_{t-i}, we can write

$$y_{t-i} = y_{t-i-1} + F(1)S_{1,t-i} + F^*(L)\varepsilon_{t-i}$$

$$= \sum_{s=1}^{t-i}[F(1)S_{1s} + F^*(L)\varepsilon_s]$$

$$= F(1)\sum_{s=1}^{t-i} S_{1s}$$

$$+ \sum_{s=1}^{t-i}[F^*(1) + (1-L)F^{**}(L)]\varepsilon_s$$

$$= F(1)S_{2,t=i} + \sum_{s=1}^{t-i} F^*(1)\varepsilon_s + \sum_{s=1}^{t-i} F^{**}(L)\Delta\varepsilon_s$$

Finally, we get

$$y_{t-i} = F(1)S_{2,t-i} + F^*(1)S_{1,t-i} + F^{**}(L)\varepsilon_{t-i}$$

Asymptotic Limits

Assuming that w_t and ε_t satisfy the conditions of the invariance principe, we can write

$$X_T(r) = T^{-\frac{1}{2}}\sum_{j=1}^{t-i} \varepsilon_j \overset{L}{\rightarrow} B(r) \; for \; \frac{t-1}{T} \leq r < \frac{t}{T}$$

Where $B(r)$ is a standard Brownian motion and

$$X_{0T}(r) = T^{-\frac{1}{2}}\sum_{j=1}^{t-i} \omega_j \overset{L}{\rightarrow} B_0(r) \; for \; \frac{t-1}{T} \leq r < \frac{t}{T}$$

Where $B_0(r)$ is a Brownian motion with covariance matrix Ω.

Let us partition B_0 (r) according to $\omega_t' = (\omega_{1t}', \ \omega_{2t}', \ \omega_{3t}')$

$$B_0(r)' = (B_{01}(r)', \ B_{02}(r)', \ B_{03}(r)')$$

Further, assume that $\forall i = 1, \ 2, \ 3, \ldots \ldots p$ and ε_{t-i} are independent.

Applying various results on Brownian motion, we obtain

1. For *i = 1, 2, …., p*

$$(a) \qquad T^{-\frac{1}{2}} \sum_{t}^{T} \Delta^2 y_{t-i} \otimes \omega_t \xrightarrow{L} N\left[0, \qquad E\left(\Delta^2 y_{t-i} \Delta^2 {y'}_{t-i}\right) \otimes E(\omega_t {\omega'}_t)\right]$$

$$* \quad E\,(\omega_t {\omega'}_t) = \Omega$$

$$* \, E\left(\Delta^2 y_{t-i} \Delta^2 {y'}_{t-i}\right) = E\left[F(L)\, \varepsilon_{t-i}\, \varepsilon_{t-i} F(L)'\right] = F(L) F(L^{-1})'$$

(b) for all s = 0, 1, 2,……

$$T^{-1} \Delta^2 {Y'}_{-i} \Delta^2 Y = T^{-1} \sum_{t=i+1}^{T} \Delta^2 y_{t-i} \Delta^2 {y'}_{t-s-i} \xrightarrow{P} E\left(\Delta^2 y_{t-i} \Delta^2 {y'}_{t-s-i}\right)$$

$$* \, E\left(\Delta^2 y_{t-i} \Delta^2 {y'}_{t-i-s}\right) = \sum_{j=0}^{\infty} f_{j+s} {f'}_j = \gamma_s$$

$$\Longrightarrow T^{-1} {\xi'}_{-1} \xi_1 = T^{-1} \sum_{t=1}^{T} \xi_{t-1} \xi_{-1} \xrightarrow{L} \Gamma$$

$$= \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_p \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{p-1} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{-p} & \gamma_{-p+1} & \cdots & \gamma_0 \end{pmatrix}$$

(c) $$T^{-1}\Delta Y'_{-1}\Delta^2 Y_{-i} = T^{-1} \quad \sum_{t=i+1}^{T} \Delta y_{t-1}\Delta^2 y'_{t-i}$$

$$= T^{-1} \sum_{t=i+1}^{T} \left[F(1)S_{1,t-1} + F^*(L)\varepsilon_{t-1}\right][F(L)\varepsilon_{t-i}]'$$

$$= F(1)T^{-1} \sum_{t=i+1}^{T} S_{1,t-1}\,[F(L)\varepsilon_{t-i}]'$$

$$+ T^{-1} \sum_{t=i+1}^{T} [F^*(L)\varepsilon_{t-1}][F(L)\varepsilon_{t-1}]'$$

$$= F(1)T^{-1} \sum_{t=i+1}^{T} S_{1,t-1}\left[F(L)L^{i-1}\varepsilon_{t=1}\right]'$$

$$+ T^{-1} \sum_{t=i+1}^{T} [F^*(L)\varepsilon_{t-1}][F(L)\varepsilon_{t-1}]'$$

$$\xrightarrow{L} F(1)\left[\int_0^1 B(r)dB(r)'\,\tilde{F}(1)' + \tilde{F}(1)' + \Gamma_{i-1}^*\right]$$

with $\tilde{F}(L) = F(L)L^{i-1}$; thus $\tilde{F}(1) = F(1)$

And $\Gamma_{i-1}^* = E([F^*(L)\varepsilon_{t-1}][F(L)\varepsilon_{t-i}]')$

$$= \sum_{j=0}^{\infty} f_{j+i-1}^* f_j'$$

$$\Rightarrow T^{-1}\Delta Y'_{-1}\Delta^2 Y_{-i} \xrightarrow{L} F(1)\int_0^1 B(r)dB(r)'\,F(1)' + \Delta i$$

$$Where\ \Delta_i = F(1)F(1)' + \sum_{j=0}^{\infty} f_{j+i-1}^* f_j'$$

$$T^{-1}\Delta Y'_{-1}\xi_{-1} = T^{-1}\sum_{t=1}^{T}\Delta y_{t-1}\xi'_{t-1}$$

$$= T^{-1}\sum_{t=1}^{T}\Delta y_{t-1}\left[\Delta^2 y'_{t-1}\ \Delta^2 y'_{t-2}\ \ \ldots.\Delta^2 y'_{t-p}\right]$$

$$\Rightarrow T^{-1}\Delta Y'_{-1}\,\xi_{-1} \xrightarrow{L} \tau'_p \otimes F(1)\int_0^1 B(r)dB(r)'F(1)' + \Delta$$

With τ_p being the unit vector of order p and $\Delta = \left[\Delta_1\ \ \Delta_2\ \ldots.\ \ \Delta_p\right]$

(d) $T^{-2}Y'_{-2}\Delta^2 Y_{-i} = T^{-2}\ \sum_{t=i+1}^{T} y_{t-2}\Delta^2 y'_{t=i}$

$$= T^{-2}\sum_{t=i+1}^{T}\left[F(1)S_{2,t-2} + F^{*}(1)S_{1,t-2} + F^{**}(L)\varepsilon_{t-2}\right][F(L)\varepsilon_{t-1}]$$

$$= F(1)T^{-2}\sum_{t}^{T} S_{2,t-2}\,[F(L)\varepsilon_{t-i}]'$$

$$+ F^{*}(1)T^{-2}\sum_{t=i+1}^{T} S_{1,t-2}[F(L)\varepsilon_{t-i}]'$$

$$+ T^{-2}\sum_{t=i+1}^{T}[F^{**}(L)\varepsilon_{t-2}][F(L)\varepsilon_{t-i}]'$$

- $T^{-2}\ \sum_{t=i+1}^{T} S_{2,t-2}\ \ [F\ (L)\ \varepsilon_{t-i}]' =$
  $T^{-2}\ \sum_{t=i+1}^{T} S_{2,t-2}\left[\ddot{F}\ \ (L)\varepsilon_{t-2}\right]'$
  $\xrightarrow{L}\int_0^1 \bar{B}\ (r)\ dB\ (r)'\ \ddot{F}\ (1)'$

With $\ddot{F}\ (L) = F\ (L) = F\ (L)L^{i-2}$, thus $\ddot{F}\ \ (1) = F\ (1)$

$$\Rightarrow T^{-2}\sum_{t=i-1}^{T} S_{2,t-2}\,[F\ (L)\varepsilon_{t-i}]' \xrightarrow{L}\int_0^1 \bar{B}\ (r)dB(r)'F(1)'$$

- $T^{-2}\ \sum_{t=i+1}^{T} S_{1,t-2}\,[F\ (L)\varepsilon_{t-i}]' =$
  $T^{-1}\ T^{-1}\ \sum_{t}^{T} S_{1,t-2}\left[\ddot{F}\ (L)\varepsilon_{t-2}\right]'$
  $\overset{L}{\rightarrow}\ 0\ \left[\int_0^1 B\ (r)dB\ (r)'\ \ddot{F}\ \ (1)'\right] = 0$

- $T^{-2}\ \sum_{t=i+1}^{T} [F^{**}\ (L)\varepsilon_{t-2}]\ [\mathrm{F}\ (\mathrm{L})\ \varepsilon_{t-i}]'$

  $= T^{-1}\,T^{-1}\ \sum_{t=i+1}^{T} [F^{**}\ (L)\varepsilon_{t-2}\,]\ [F(L)\varepsilon_{t-i}]'$
  $\overset{L}{\rightarrow}\ 0\ \sum_{j=0}^{\infty} f_{j+i-2}^{**}\ f_j = 0$

Combining the above there points, we can write

$T^{-2}\ Y'_{-2}\,\Delta^2\ Y_{-1}\ \overset{L}{\rightarrow} F(1)\int_0^1 \overline{B}\ (r)dB(r)'F(1)'$

$\Longrightarrow T^{-2}\ Y'_{-2}\xi_{-1} \overset{L}{\rightarrow} \tau_p\ \otimes F(1)\int_0^1 \overline{B}\ (r)dB(r)'F(1)'$

2. $(a)T^{-2}\ \ \Delta\,Y'_{-1}\ \ \Delta\,Y_{-1}\ =\ T^{-2}\ \sum_{t=1}^{T} \Delta\,y_{t-1}\ \Delta\ \ y'_{-1}$

$= T^{-2}\sum_{t=1}^{T}\left[F(1)S_{1,t-1} + F^{*}(L)\varepsilon_{t-1}\right]\left[F(1)S_{1,t-1} + F^{*}\ (L)\varepsilon_{t-1}\right]'$

$= F(1)T^{-2}\ \sum_{t=1}^{T} S_{1,t-1}\ S'_{1,t-1}\ F\ (1)' +$
$T^{-1}\,F\ (1)\ T^{-1}\ \ \sum_{t=1}^{T} S\ _{1,t-1}\ [F^{*}\ (L)\varepsilon_{t-1}]'$

$+\ T^{-1}\ T^{-1}\sum_{t=1}^{T}[F^{*}\ (L)\varepsilon_{t-1}]'\ +\ T^{-1}\,T^{-1}\ \sum_{t=1}^{T}[F^{*}\ (L)\varepsilon_{t-1}\ ]\ \ [F^{*}\ (L)\varepsilon_{t-1}]'$

The terms $T^{-1}\ \sum_{t=1}^{T}\ S_{1,t-1}\ [F^{*}\ \ (L)\varepsilon_{t-1}]'\,, T^{-1}\ \ \sum_{t=1}^{T}[\,F^{*}\ \ (L)\varepsilon_{t-1}][\,F^{*}\,(L)\varepsilon_{t-1}]'$

and $T^{-1}\sum_{t=1}^{T}[\,F^{*}\ \ (L)\varepsilon_{t-1}]\,[F^{*}\,(L)\varepsilon_{t-1}]'$ have finite limits and T^{-1} tends to zero, thus

$$T^{-2}\Delta\, Y'_{-1}\,\Delta\, Y_{-1} \cong^{as} F\,(1)\, T^{-2} \sum_{t=1}^{T} S_{1,t-1}S'_{1,t-\ 1}F(1)$$

$$\xrightarrow{L} F(1) \int_0^1 B(r)B(r)'dr\, F(1)'$$

$$\Rightarrow T^{-2}\Delta\ \ Y'_{-1}\ \Delta Y_{-1} \xrightarrow{L} F(1) \int_0^1 B(r)\, B(r)'\ \mathrm{dr}\ \mathrm{F}(1)'$$

$(b)\ \ T^{-3}\Delta\ \ Y'_{-1}\, Y_{-2} = T^{-3}\sum_t^T \Delta y_{t-1}\, y'_{t-2}$

$$= T^{-3} \sum_t^T \left[F(1)S_{1,t-1}F^*(L)\varepsilon_{t-1}\ \ \right]\left[F(1)S_{2,t-2} + F^*(1)S_{1,t-2} + F^{**}(L)\varepsilon_{t-2}\right]$$

Similarly to (a), we get

$(b)\ \ \ T^{-3}\Delta\ \ Y'_{-1}Y_{-2} \cong^{as} \mathrm{F}(1)T^{-3}\sum_{t=1}^T S_{1,t-1}\, S'_{\,2,t-2}\mathrm{F}(1)'$

$$\xrightarrow{L}\ \mathrm{F}(1) \int_0^1 B(r)\mathrm{B}(r)'\mathrm{dr}\ \mathrm{F}(1)'$$

$$\Rightarrow T^{-3}\Delta\ \ Y'_{-1}Y_{-2} \xrightarrow{L} F(1) \int_0^1 B(r)\, B(r)'\ \mathrm{dr}\ \mathrm{F}(1)'$$

$(\mathrm{c}\,)\ T^{-1}Y'_{-1}Y_{-1} = T^{-1}\sum_t^T y_{t-2}y'_{t-2}$

$$= T^{-4}\sum_{t=1}^T\left[F(1)S_{2,t-2} + F^*(1)S_{1,t-2} + F^{**}(L)\varepsilon_{t-2}\ \ \right]' \times \left[F(1)S_{2,t-2} + F^*(1)S_{1,t-2} + F^{**}(L)\varepsilon_{t-2}\ \ \right]'$$

For the same reasons as in 2. $(a)$and 2.(b), we obtain

$$T^{-4}Y'_{\ -1}Y'_{\ -1} = T^{-4}\sum_{t=1}^T y_{t-2}\quad y_{t-2} \overset{as}{\cong} \mathrm{F}(1)T^{-4}\sum_t^T S_{2,t-2}\, S'_{\,2,t-2}\mathrm{F}(1)'$$

$$\xrightarrow{L}\ F(1) \int_0^1 \bar{B}(r)\, \bar{B}(r)'\ \mathrm{dr}\ \mathrm{F}(1)'$$

$$T^{-4}Y'_{\ -2}Y_{-2} \xrightarrow{L}\ F(1) \int_0^1 B(r)\, B(r)'\ \mathrm{dr}\ \mathrm{F}(1)'$$

3. $(a)\ \ T^{-1}(\Delta Y'_{-1} \otimes I_n)w = T^{-1}\sum_{t=1}^T \Delta\, y_{t-1} \otimes w_t$

$$=\text{vec}\left[T^{-1}\sum_{t=1}^{T} w_t \Delta y'_{t-1}\right] = \text{vec}[(T^{-1}\Delta y_{t-1} w'_t)']$$

- $T^{-1} \sum_{t=1}^{T} \Delta y_{t-1} w'_t = T^{-1} \sum_{t=1}^{T} \left[F(1) S_{1,t-1} + F^*(L)\, \varepsilon_{t-1}\right] w'_t$

$$= F(1)\, T^{-1} \sum_{t-1}^{T} S_{1\,t-1}\, w'_t + T^{-1} \sum_{t=1}^{T} [F^*(L)\varepsilon_{t-1} w'_t]$$

Using the independence between w_t and ε_{t-1}

$$T^{-1} \sum_{t=1}^{T} S_{1,t-1}\, w'_t \xrightarrow{L} \int_0^1 B(r)\, dB_0(r)'$$

and $T^{-1} \sum_{t=1}^{T} (F^*(L)\, \varepsilon_{t-1}) w'_t \xrightarrow{L} 0$

Hence $T^{-1} \sum_{t=1}^{T} \Delta y_{t-1}\; w'_t \xrightarrow{L} F(1) \int_0^1 B(r) d\, B_0(r)'$

and

Vec $(T^{-1} \sum_{t=1}^{T} w_t\; \Delta y'_{t-1}) \xrightarrow{L} vec\, [\int_0^1 dB_0(r)\; B(r)' F(1)']$

$$= \int_0^1 F(1) B(r) \otimes dB_0(r)$$

$$\Rightarrow T^{-1} (\Delta Y'_{-1} \otimes I_n) w \xrightarrow{L} [F(1) \otimes I_n] \int_0^1 B(r) \otimes d\, B_0(r)$$

(b) $T^{-2} (Y'_{-2} \otimes I_n) w = T^{-2} \sum_{t=1}^{T} y_{t-2} \otimes w_t = vec\left[\left(T^{-2} \sum_{t=1}^{T} y_{t-2w_t{}'}\right)'\right]$

- $T^{-2} \sum_{t=1}^{T} y_{t-2} w'_t = T^{-2} \sum_{t=1}^{T} \left[F(1) S_{2,t-2} + F^*(1) S_{1,t-1} + F^{**}(L)\varepsilon_{t-2}\right] w'_t$

$$= F(1)T^{-2}\sum_{t=1}^{T} S_{2,t-2}w'_t + T^{-1}F^*(1)T^{-1}\sum_{t=1}^{T} S_{1,t-2}w'_t$$
$$+T^{-2}\sum_{t=1}^{t}[F^{**}(L)\varepsilon_{t-2}]\,w'_t$$

$$\xrightarrow{L} F(1)\int_0^1 \bar{B}(r)d\,B_0\ (r)' + 0\int_0^1 B(r)d\,B_0\ (r)' + 0$$

Thus $T^{-2}\sum_{t=1}^{T} w'_t y'_{t-2} \xrightarrow{L} \int_0^1 d\,B_0\ (r)'\,\bar{B}(r)\,F(1)$

Therefore

$\mathrm{Vec}\left[T^{-2}\sum_{t=1}^{T} w'_t y'_{t-2}\right] \xrightarrow{L} \mathrm{vec}\left[\int_0^1 d\,B_0\ (r)\quad \bar{B}(r)'\,F(1)'\right] = \int_0^1 F\ (1)\bar{B}(r) \otimes d\,B_0(r)$

$$\Longrightarrow\ T^{-2}(Y'_{-2}\otimes I_n)w \xrightarrow{L} [F(1)\otimes I_n]\int_0^1 \bar{B}(r)\otimes dB_0(r)$$

From(1), (2) *and* (3) *above, the following result can be derived* :

$*$ ***limit of*** $\boldsymbol{\psi_{11T}}$

$$T^{-2}\Delta Y'_{-1}M_\xi\Delta Y_{-1}$$
$$= T^{-2}\Delta Y'_{-1}\ \Delta Y_{-1}$$
$$- T^{-1}T^{-1}\Delta Y'_{-1}\xi_{-1}(T^{-1}\xi_{-1}\xi_{-1})^{-1}T^{-1}\xi'_{-1}\ \Delta Y_{-1}$$

Knowing that $T^{-1}\xi_{-1}\xi_{-1}$, $T^{-1}\Delta Y'_{-1}\xi_{-1}$ and $T^{-1}\xi'_{-1}\ \Delta Y_{-1}$ *have finite limits and that* T^{-1} *goes to zero, we get*

$$T^{-2}\Delta Y'_{-1}M_\xi\Delta Y_{-1} \overset{as}{\simeq} T^{-2}\Delta Y'_{-1}\ \Delta Y_{-1}$$

$$\Longrightarrow\ \psi_{11T} \overset{as}{\simeq} T^{-2}\Delta Y'_{-1}\ \Delta Y_{-1}\otimes\Omega^{-1} \xrightarrow{L} F(1)\int_0^1 B(r)B(r)'dr\,F(1)'\otimes\Omega^{-1}$$
$$\equiv \psi_{11}$$

- **Limit of $\boldsymbol{\psi_{12T}}$**

$$T^{-3}\Delta Y'_{-1}M_{\xi}\Delta Y_{-2}$$
$$= T^{-3}\Delta Y'_{-1}Y_{-2}$$
$$- T^{-1}T^{-1}\Delta Y'_{-1}\xi_{-1}(T^{-1}\xi'_{-1}\xi_{-1})^{-1}T^{-2}\xi'_{-1}\,\Delta Y_{-2}$$

Using the same arguments as before and the previous result which implies that $T^{-3}\xi'_{-1}\,\Delta Y_{-2}$ has finite limit, we have

$$T^{-3}\Delta Y'_{-1}M_{\xi}\Delta Y_{-2}\overset{as}{\simeq} T^{-3}\Delta Y'_{-1}Y_{-2}$$

$$\Longrightarrow\ \psi_{12T}\overset{as}{\simeq} T^{-3}\Delta Y'_{-1}Y_{-2}\otimes\Omega^{-1}$$
$$\overset{L}{\to} F(1)\int_0^1 B(r)B(r)'dr\,F(1)'\otimes\Omega^{-1}\equiv\psi_{12}$$

- **Limit of $\boldsymbol{\psi_{21T}}$**

$$\psi_{21T} = \psi'_{12T}\overset{L}{\to}\psi'_{12} = F(1)\int_0^1 B(r)B(r)'dr\,F(1)'\otimes\Omega^{-1}\equiv\psi_{21}$$

- **Limit of $\boldsymbol{\psi_{22T}}$**

$$T^{-4}Y'_{-2}M_{\xi}Y_{-2}$$
$$= T^{-4}Y'_{-2}Y'_{-2}$$
$$- T^{-1}T^{-2}Y'_{-2}\xi_{-1}(T^{-1}\xi'_{-1}\xi_{-1})^{-1}T^{-2}\xi'_{-1}\,\Delta Y_{-2}$$

As $T^{-1}\xi'_{-1}\xi_{-1}$ *and* $T^{-2}Y'_{-2}\xi_{-1}$ have finite limits by earlier results and T^{-1} goes to zero, so

$$T^{-4}Y'_{-2}M_{\xi}Y_{-2}\overset{as}{\simeq} T^{-4}Y'_{-2}Y'_{-2}$$

$$\Longrightarrow \psi_{22T}\overset{as}{\simeq}T^{-4}Y'_{-2}Y'_{-2}\otimes\Omega^{-1}$$
$$\overset{L}{\to} F(1)\int_0^1 B(r)B(r)'dr\,F(1)'\otimes\Omega^{-1}\equiv\psi_{22}$$

- **Limit of $\boldsymbol{\rho_{11T}}$**

$$T^{-1}\left(\Delta Y'_{-1}\,M_{\xi}\otimes I_n\right)w$$

$$= T^{-1}\ (\Delta\, Y'_{-1}\ \otimes I_n)w - T^{-\frac{1}{2}}[T^{-1}\ \Delta\, Y'_{-1}\ \xi_{-1}\ (T^{-1}\,\xi'_{-1}\,\xi_{-1})^{-1}\ \otimes\ I_n]\,T^{-\frac{1}{2}}(\xi'_{-1} \otimes\ I_n)w$$

In a similar way to the other points, $T^{-\frac{1}{2}}(\xi'_{-1}\ \otimes\ I_n)\,w$ has a finite and $T^{-\frac{1}{2}}$ tends to zero, so we have

$$T^{-1}\left(\Delta\, Y'_{-1}\ M_\xi\ \otimes\ I_n\right)w\ \cong^{as}\ T^{-1}\ (\Delta\, Y'_{-1}\ \otimes\ I_n)w$$

$$\Longrightarrow\ \rho_{11T} = (I_n\ \otimes \Omega^{-1})\left(T^{-1}\Delta Y'_{-1}\, M_\xi \otimes\ I_n\right)w \cong^{as} (I_n \otimes \Omega^{-1})\ T^{-1}\ (\Delta\, Y'_{-1}\ \otimes\ I_n)\ w$$

$$\xrightarrow{L} (I_n\ \otimes\ \Omega^{-1})[F\ (1)\ \otimes\ I_n]\int_0^1 B(r) \otimes dB_0\,(r)$$

$$= (F\ (1) \otimes\ \Omega^{-1})\ \int_0^1 B\ (r) \otimes d\ B_0\ (r)\ \equiv\ \rho_{11}$$

- **Limit of $\boldsymbol{\rho_{22T}}$**

Using similar explanations as for earlier, it is easily seen that

$$T^{-2}\ \left(\ Y'_{-1}\ M_\xi\ \otimes\ I_n\right)w$$

$$= T^{-2}\,(Y'_{-2} \otimes\ I_n)w - T^{-\frac{1}{2}}\ [T^{-2}\ Y'_{-1}\ \xi_{-1}\ (T^{-1}\ \xi'_{-1}\ \xi_{-1})^{-1} \otimes I_n](T^{-\frac{1}{2}}Y'_{-1}\ \otimes\ I_n)\mathrm{w}$$

$$\cong^{as}\ T^{-2}\,(Y'_{-2} \otimes\ I_n)w$$

$$\Longrightarrow\ \rho_{22T} = (I_n\ \otimes\ \Omega^{-1})\left(T^{-2}Y'_{-2}\, M_\xi \otimes\ I_n\right)w\ \cong^{as}\ (I_n\ \otimes\ \Omega^{-1})\ T^{-2}\,(Y'_{-2}\ \otimes\ I_n)w$$

$$\xrightarrow{L} (I_n\ \otimes\ \Omega^{-1})[F\ (1)\ \otimes\ I_n\ ]\int_0^1 \overline{B}\ (r) \otimes dB_0\,(r)$$

$$= [F\ (1) \otimes\ \Omega^{-1}\,]\int_0^1 \overline{B}\ \ (r)\ \otimes d\ B_0\,(r) \equiv\ \rho_{22}$$

Combining all these results, we derive that

1. $T\left(\widehat{\Pi_1} - \Pi_1\right) =\ \Psi_{1T}^{-1}\,\rho_{1T}$

$$\Psi_{1T} = R_1' \Psi_{11T} R_1 - R_1' \Psi_{12T} R_2 [R_2' \Psi_{22T} R_2]^{-1} R_2' \Psi_{21T} R_1$$

$$\overset{L}{\rightarrow} R_1' \Psi_{11} R_1 - R_1' \Psi_{12T} R_2 [R_2' \Psi_{22T} R_2]^{-1} R_2' \Psi_{21T} R_1 \equiv \Psi_1$$

$$\rho_{1T} = \rho_1' \rho_{11T} - R_1' \Psi_{12T} R_2 [R_2' \Psi_{22T} R_2]^{-1} R_2' \rho_{22T}$$

$$\overset{L}{\rightarrow} R_1' \rho_{11} - R_1' \Psi_{12} R_2 [R_2' \Psi_{22} R_2]^{-1} R_2' \rho_{22} \equiv \rho_1$$

$$\Longrightarrow T\left(\widehat{\Pi}_1 - \Pi_1\right) \overset{L}{\rightarrow} \Psi_1^{-1} \rho_1 \qquad (A2.1)$$

2. $T^2 \left(\widehat{\Pi}_2 - \Pi_2\right) = \Psi_{2T}^{-1} \rho_{2T}$

$$\Psi_{2T} = R_2' \Psi_{22T} R_2 - R_2' \Psi_{21T} R_1 [R_1' \Psi_{11T} R_1]^{-1} R_1' \Psi_{12T} R_2$$

$$\overset{L}{\rightarrow} R_2' \Psi_{22} R_2 - R_2' \Psi_{21} R_1 [R_1' \Psi_{11} R_1]^{-1} R_1' \Psi_{12} R_2 \equiv \Psi_2$$

$$\rho_{2T} = R_2' \rho_{22T} - R_2' \Psi_{21T} R_1 [R_1' \Psi_{11T} R_1]^{-1} R_1' \rho_{11T}$$

$$\overset{L}{\rightarrow} R_2' \rho_{22} - R_2' \Psi_{21} R_1 [R_1' \Psi_{11} R_1]^{-1} R_1' \rho_{11} \equiv \rho_2$$

$$\Longrightarrow T^2 \left(\widehat{\Pi}_2 - \Pi_2\right) \overset{L}{\rightarrow} \Psi_2^{-1} \rho_2 \qquad (A2.2)$$

Regrouping these two results; we obtain

$$\begin{pmatrix} T\,(\widehat{\Pi}_1 - & \Pi_1) \\ T^2\,(\widehat{\Pi}_2 - & \Pi_2) \end{pmatrix} \overset{L}{\rightarrow} \begin{pmatrix} R_1' \Psi_{11} R_1 & R_1' \Psi_{12} R_2 \\ R_2' \Psi_{22} R_1 & R_2' \Psi_{22} R_2 \end{pmatrix} \begin{pmatrix} R_1' \rho_{11} \\ R_2' \rho_{22} \end{pmatrix} \qquad (A2.3)$$

- $\begin{pmatrix} R_1' \Psi_{11} R_1 & R_1' \Psi_{12} R_2 \\ R_2' \Psi_{21} R_1 & R_2' \Psi_{22} R_2 \end{pmatrix} = \begin{pmatrix} R_1' & 0 \\ 0 & R_2' \end{pmatrix} \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$

Now

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} F(1) \int_0^1 B(r)\, B(r)'\, dr\, F(1)' \otimes \Omega^{-1} & F(1) \int_0^1 B(r)\, \bar{B}(r)'\, dr\, F(1) \otimes \Omega^{-1} \\ F(1) \int_0^1 \bar{B}(r) B(r)'\, dr\, F(1)' \otimes \Omega^{-1} & F(1) \int_0^1 \bar{B}(r) \bar{B}(r)'\, dr\, F(1)' \otimes \Omega^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} F(1) \otimes I_n & 0 \\ 0 & F(1) \otimes I_n \end{pmatrix}$$

$$[\begin{pmatrix} \int_0^1 B(r)B(r)'\,dr & \int_0^1 B(r)\bar{B}(r)'\,dr \\ \int_0^1 \bar{B}(r)\,B(r)'\,dr & \int_0^1 \bar{B}(r)\bar{B}(r)'\,dr \end{pmatrix} \otimes \Omega^{-1}]$$

$$\times \begin{pmatrix} F(1)' \otimes I_n & 0 \\ 0 & F(1)' \otimes I_n \end{pmatrix}$$

Hence

$$\begin{pmatrix} R_1' \Psi_{11} R_1 & R_1' \Psi_{12} R_2 \\ R_2' \Psi_{21} R_1 & R_2' \Psi_{22} R_2 \end{pmatrix} = J\,(\int_0^1 \boldsymbol{\mathcal{B}}(r)\,\boldsymbol{\mathcal{B}}(r)'\,dr \otimes \Omega^{-1})J'$$

Where $J = \begin{pmatrix} R_1'(F(1) \otimes I_n) & 0 \\ 0 & R_2'(F(1) \otimes I_n) \end{pmatrix}$ and $\boldsymbol{\mathcal{B}}(\boldsymbol{r})' = [B(r)' \quad \bar{B}(r)']$

- $\begin{pmatrix} R_1' \rho_{11} \\ R_2' \rho_{22} \end{pmatrix} = \begin{pmatrix} R_1' & 0 \\ 0 & R_2' \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{22} \end{pmatrix}$

$$= \begin{pmatrix} R_1' & 0 \\ 0 & R_2' \end{pmatrix} \begin{pmatrix} [F(1) \otimes \Omega^{-1}] & \int_0^1 B(r) \otimes dB_0(r) \\ [F(1) \otimes \Omega^{-1}] & \int_0^1 \bar{B}(r) \otimes d\,B_0(r) \end{pmatrix}$$

$$= \begin{pmatrix} R_1' & 0 \\ 0 & R_2' \end{pmatrix} \begin{pmatrix} (F(1) \otimes I_n) & 0 \\ 0 & (F(1) \otimes I_n) \end{pmatrix} \begin{pmatrix} I_n \otimes \Omega^{-1} & 0 \\ 0 & I_n \otimes \Omega^{-1} \end{pmatrix} \begin{pmatrix} \int_0^1 B(r) \otimes d\,B_0(r) \\ \int_0^1 \bar{B}(r) \otimes dB_0(r) \end{pmatrix}$$

$$= J\,[I_{2n} \otimes \Omega^{-1}] \int_0^1 \boldsymbol{\mathcal{B}}(r) \otimes B_0(r)$$

Where it is assumed that $B_0(r)$ is independent of $\boldsymbol{\mathcal{B}}(r)$. This is due to the independence between w_t et $\varepsilon_{t-i}, i = 1, 2, 3 \ldots$.

Finally $\begin{pmatrix} T\,(\widehat{\Pi}_1 - & \Pi_1) \\ T^2\,(\,\widehat{\Pi}_2 - & \Pi_2\,) \end{pmatrix} \overset{L}{\rightarrow} [J\,\left(\int_0^1 \boldsymbol{\mathcal{B}}\,(r)\boldsymbol{\mathcal{B}}(r)'\,dr\,\otimes\,\,\Omega^{-1}\right)J']^{-1}$

$$\times J\,(I_{2n}\,\otimes\,\Omega^{-1})\int_0^1 \boldsymbol{\mathcal{B}}\,(r)\,\otimes dB_0\,(r)$$

Using a discussion similar to *Phillips (1991)* we obtain the following mixed normal distribution

$$\begin{pmatrix} T\,(\widehat{\Pi}_1 - & \Pi_1) \\ T^2\,(\,\widehat{\Pi}_2 - & \Pi_2\,) \end{pmatrix} \overset{L}{\rightarrow} \int_{G>0} N\,(0,[J\,(G\,\otimes\,\,\Omega^{-1}\,)\,J'\,]^{-1})\,dP(G)$$

Where $G = \int_0^1 \boldsymbol{\mathcal{B}}(r)B\,(r)'\,dr$ and P is a probability measure associated with G.

In case is G known then $P(G) = 1$.

Annex 3: The estimating equation

Let us recall our triangular system

$$\begin{cases} \Delta^2\,y_{1t} = u_{1t} \\ \Delta\,y_{2t} = \theta_1\,\Delta\,y_{1t} + u_{2t} \\ y_{3t} = \theta_2\,y_{2t} + \theta_3 y_{1t} + \theta_4\Delta y_{1t} + u_{3t} \end{cases}$$

We are concerned with the estimation of the third equation knowing θ_1. Taking this information into account, we can do the following transformations

$$\Delta^2 y_{2t} = \Delta y_{2t} - \Delta y_{2,t+1} = D\,_1(L)\Delta^2 y_{1t} + \theta_1\,\Delta y_{1t} + v_{2t} - \Delta y_{2,t-1}\ (1)$$

Since $\Delta^2\,y_{1t} = v_{1t}$ therefore $\Delta\,y_{1t} = \Delta\,y_{1,t-1} + v_{1t}$ and writing

$$D_1\,(L) = d_{10} + LD_{11}\,(L)$$

We obtain

$$D_1(L)\Delta^2\,y_{1t} = d_{10}\,\Delta^2\,y_{1t} + D_{11}(L)\Delta^2 y_{1,t-1} = d_{10}\,v_{1t} + D\,_{11}\,(L)\Delta^2\,y_{1,t-1}\ (2)$$

With $D_{11}\,(1) < \infty$

$\Delta^2 y_{2t} = D_{11}(L)\Delta^2 y_{2,t-1} - (\Delta y_{2,t-1} - \theta_1 \Delta y_{1,t-1}) + (d_{10} + \theta_1)v_{1t} + v_{2t}$ (3)

Writing $\Delta y_{2t} - \theta_1 \Delta y_{1t} = D_1(L)\Delta^2 y_{1t} + v_{2t}$ for $t-1$, and substituting in (3), we have

$$\Delta^2 y_{2,t} = A_{1*}(L)\Delta^2 y_{1,t-1} + w_{2t} \quad (4)$$

With $A_{1*}(L) = D_{11}(L) - D_1(L)$ and $w_{2t} = (d_{10} + \theta_1)v_{1t} + (1-L)v_{2t}$

From equation (4) , we derive

$$y_{2t} = 2\Delta y_{2,t-1} + y_{2,t-2} + A_{1*}(L)\Delta^2 y_{1,t-1} + (1-L)v_{2t}$$

$$= 2[\Delta y_{2,t-1} - \theta_1 \Delta y_{1,t-1}] + 2\theta_1 \Delta y_{1,t-1} + y_{2,t-2} + A_{1*}(L)\Delta^2 y_{1,t-1} + (d_{10} + \theta_1) + (1-L)v_{2t}$$

$$= 2[D_1(L)\Delta^2 y_{1,t-1} + v_{2,t-1}] + 2\theta_1 \Delta y_{1,t-1} + y_{2,t-2} + A_{1*}(L)\Delta^2 y_{1,t-1} + (d_{10} + \theta_1)v_{1t} + (1-L)v_{2t}$$

$$\Longrightarrow y_{2t} = D_{1*}(L)\Delta^2 y_{1,t-1} + 2\theta_1 \Delta y_{1,t-1} y_{2,t-2} + (d_{10} + \theta_1)v_{1t} + (1+L)v_{2t}$$

With $D_{1*}(L) = 2D_1(L) + A_{1*}(L) = D_1(L) + D_{11}(L)$

Using all these results in the third equation of our system yields

$$y_{3t} = D_2(L)\Delta^2 y_{1t} + D_3(L)[D_1(L)\Delta^2 y_{1t} + v_{2t}]$$

$$+ \theta_2 [D_{1*}(L)\Delta^2 y_{1,t-1} + 2\theta_1 \Delta y_{1,t-1} + y_{2,t-2} + (d_{10} + \theta_1)v_{1t} + (1+L)v_{2t}]$$

$$+\theta_3[2\Delta y_{1,t-1} + y_{1,t-2} + v_{1t}] + \theta_4(\Delta y_{1,t-1} + v_{3t})$$

As $D_2(L)\Delta^2 y_{1t} = [d_{20} + LD_{21}(L)]\Delta^2 y_{1t} = d_{20}v_{1t} + D_{21}(L)\Delta^2 y_{1,t-1}$

And $D_1(L)\Delta^2 y_{1t} = [d_{10} + LD_{11}(L)]\Delta^2 y_{1t} = d_{10} v_{1t} + D_{11}(L)\Delta^2 y_{1,t-1}$

We have

$$y_{3t} = d_{20}\, v_{21} + D_{21}\ (L)\Delta^2\, y_{1,t-1} + D_3\,(L)D_{10}\, v_{1t} + D_3\,(L)D_{11}\,(L)\Delta^2\, y_{1,t-1}$$

$$+ D_3\,(L)v_{2t} + \theta_2\, D_{1*}\,(L)\,\Delta^2\, y_{1,t-1} + 2\theta_2\,\theta_1\,\Delta\, y_{1,t-1} + \theta_2\, y_{2,t-2}$$

$$+ \theta_2\,[(d_{10} + \theta_1)v_{1t} + (1 + L)\, v_{2t}] + 2\theta_3\,\Delta\, y_{1,t-1} + \theta_3\, y_{1,t-2} + \theta_2\, v_{1t}$$

$$+ \theta_4\,\Delta\, y_{1,t-1} + \theta_4\, v_{1t} + v_{3t}$$

$$y_{3t} = A_{2*}\,(L)\Delta^2\, y_{1,t-1} + \delta\,\Delta\, y_{1,t-1} + \theta_3\, y_{1,t-1} + \theta_3\, y_{1,t-2} + \theta_2\, y_{2,t-2} + w_{3t}$$

With $A_{2*}(L) = D_{21}(L) + D_3(L)D_{11}(L) + \theta_2\, A_2\,(L)$

$\delta = 2\,\theta_2\,\theta_1 + 2\,\theta_3 + \theta_4$

$$w_{3t} = [d_{20} + D_3\ (L)d_{10} + \theta_2\, d_{10} + \theta_2\theta_1 + \theta_3 + \theta_4]v_{1t} + [D_3\ (L) + \theta_2\ (1 + L)v_{\ 2t} + v_{3t}]$$

Further rearrangements :

1. $A_3\,(L)\Delta^2\, y_{1,t-1} = A_{30}\, y_{1,t-1} + A_{31}\,\Delta^2\, y_{1,t-2} + \cdots + A_{3q}\,\Delta^2\, y_{1,t-1+q} = \Theta_0\, z_{0t}$

Where $\Theta_0 = [A_{30}A_{31}\ \ldots\ldots A_{3q}]$ with q being a finite positive integer

and $z'_{0t} = \left[\Delta^2 y'_{\ 1,t-1}\Delta^2 y_{1,t-2\ldots\ldots}\,\Delta^2\, y'_{1,t-1+q}\right]$

$z_{0t} \backsim I(0)$

2. $\delta\,\Delta\, y_{1,t-1} = \Theta_1 z_{1t}$

Where $\Theta_1 = \delta$ and $z'_{1t} = \Delta y'_{1,t-1}$

$z_{1t} \backsim I(1)$

3. $\theta_3 y_{1,t-2} + \theta_2 y_{2,t-2} = \Theta_2\, z_{2t}$

Where $\Theta_2 = [\theta_3\ \ \theta_2]$ *and* $z'_{2t} = \left(y'_{1,t-2}\ \ \ y'_{2,t-2}\right)$

$z_{2t} \backsim I(2)$

Thus we obtain equation (24):

$$y_{3t} = \Theta_0 z_{0t} + \Theta_1 z_{1t} + \Theta_2 z_{2t} + w_{3t} = \Theta z_t + w_{3t}$$

Where $\Theta = [\Theta_0 \ \ \Theta_1 \ \ \Theta_2]$ and $z_t' = (z_{0t}' \ \ z_{1t}' \ \ z_{2t}')$

Note that y_{3t}, z_{it} and w_{3t} are respectively of order $(n_3 \times 1)$, $(k_i \times 1)$ and $(n_3 \times 1)$;

therefore Θ and z_t are $(n_3 \times k)$ and $(k \times 1)$ respectively.

Where $k = \sum_{i=0}^{2} k_i$

Annex 4: Asymptotic distribution of OLS

Premultiplying by $(\Gamma_T \otimes I_{n_3})$, we have

$$\Gamma_T \otimes I_{n3}\left[vec\left(\widehat{\Theta}_{ols}\right) - vec(\Theta)\right]$$

$$= [\left(\Gamma_T^{-1} \otimes I_{n_3}\right)\left(\sum_{t=1}^{T} z_t z_t' \otimes I_{n_3}\right)\left(\Gamma_T^{-1} \otimes I_{n_3}\right)]^{-1}\left(\Gamma_T^{-1} \otimes I_{n_3}\right)\sum_{t=1}^{T} z_t \otimes w_{3t}$$

$$= [\Gamma_T^{-1} \sum_{t=1}^{T} z_t z_t' \Gamma_T^{-1} \otimes I_{n_3}]^{-1} \sum_{t=1}^{T} \Gamma_T^{-1} z_t \otimes w_{3t}$$

Let $Q_T = \Gamma_T^{-1} z_t z_t' \Gamma_T^{-1} =$

$$\begin{pmatrix} T^{-1} \sum_{t=1}^{T} z_{0t} z_{0t}' & T^{\frac{3}{2}} \sum_{t=1}^{T} z_{0t} z_{1t}' & T^{-\frac{5}{2}} \sum_{t=1}^{T} z_{0t} z_{2t}' \\ T^{-\frac{3}{2}} \sum_{t=1}^{T} z_{1t} z_{0t}' & T^{-2} \sum_{t=1}^{T} z_{1t} z_{1t}' & T^{-3} \sum_{t=1}^{T} z_{1t} z_{2t}' \\ T^{-\frac{5}{2}} \sum_{t=1}^{T} \sum_{t=1}^{T} z_{2t} z_{0t}' & T^{-3} \sum_{t=1}^{T} z_{2t} z_{1t}' & T^{-4} \sum_{t=1}^{T} z_{2t} z_{2t}' \end{pmatrix}$$

And $\varphi_T = \sum_{t=1}^{T} \Gamma_T^{-1} z_t \otimes v_{3t} = \begin{pmatrix} T^{-\frac{1}{2}} \sum_{t=1}^{T} z_{0t} \otimes w_{3t} \\ T^{-1} \sum_{t=1}^{T} z_{1t} \otimes w_{3t} \\ T^{-2} \sum_{t=1}^{T} z_{2t} \otimes w_{3t} \end{pmatrix}$

Let us express z_{0t}, z_{1t} and z_{2t} in terms of the canonical base $\{\varepsilon_t,\ S_{1t},$ $S_{2t}\}$ with

$S_{1t} = \sum_{s=1}^{T} \varepsilon_{s,} S_{2t} = \sum_{s=1}^{T} s_1$ and $\varepsilon_t = S_{1t} = S_{2t} = 0 \ \forall\ t \leq 0$

1. $z'_{0t} = \left[\Delta^2\, y'_{1,t-1}\, \Delta^2 y_{1,t-2}, \ldots..\Delta^2\, y'_{1,t-1-q}\right]$

Knowing $\Delta^2\, y_{1t} = u_{1t} = F_1\ (L)\varepsilon_t$, recurrence yields

$\triangle^2\, y_{1,t-1-j} = F_1(L)L^j\, \varepsilon_{t-1} \quad for\ \ j = 0,1,2, \ldots., q.$

Therefore z_{0t} can be written as

$$z_{0t} = K_0\ (L)\varepsilon_t$$

Where $K_0\ (L)' = [F_1\ (L)'\ \ LF_1\ (L)' \ldots . L^q\ F_1\ (L)']$

2. $z_{1t} = \Delta\, y_{1,t-1}$

But $\Delta^2\ y_{1,t} = F_1\ (L)_{\varepsilon_t}$ implies $\Delta\, y_{1t} = \Delta\, y_{1,t-1}\, F_1\ (L)_{\varepsilon_t}$ and recurrence enables us to write

$$\Delta\, y_{1t} = \sum_{s=1}^{t} F_1\ (L)\varepsilon_s$$

Substituting $F_1\,(L) + (1+L)F_1^*(L)$ for $F_1(L)$ we have

$$\Delta\, y_{1t} = F_1\,(1) \sum_{s=1}^{T} \varepsilon_s + \sum_{s=1}^{t} F_1^*\ (L)\Delta\, \varepsilon_s$$

$$as \sum_{s=1}^{t} F_1^*\ (L)\Delta\, \varepsilon_s = \sum_{s=1}^{t}[\sum_{j=0}^{\infty} f_{1j}^*\, \Delta\, \varepsilon_{t-j}\,] = \sum_{j=0}^{\infty} f_{1j}^* \varepsilon_{t-j} = F_1^*\ (L)\ \varepsilon_t$$

We have $\Delta\, y_{1,t-1} = F_1\ (1)S_{1,t-1} + F_1^{\ *}\,(L)\varepsilon_{t-1}$

Thus $\Delta\, y_{1,t-1} = F_1\,(1)S_{1,t-1} + F_1^{\ *}\,(L)\ \varepsilon_{t\,-1}$

$$\Longrightarrow\ z_{1t} = F_1^*\ (L)\varepsilon_{t-1} + F_1\ (1)S_{1,t-1}$$

3. $z_{2t} = \binom{y_{1,t-2}}{y_{2,t-2}}$

- $\Delta y_{1t} = F_1(1)S_{1t} + F_1^*(L)\varepsilon_t$
  $\Rightarrow y_{1t} = y_{1,t-1} + F_1(1)S_{1t} + F_1^*(L)\varepsilon_t$

By recurrence,

$$y_{1t} = \sum_{s=1}^{t}[F_1(1)S_{1t} + F_1^*(L)\varepsilon_t]$$

Substituting $F_1^*(L) by\, F_1^*(1) + (1-L)F_1^{**}(L)$, we get

$$y_{1t} = F_1(1)\sum_{s=1}^{t} S_{1t} + \sum_{s=1}^{t} s_{1t} + \sum_{s=1}^{t}[F_1^*(1)\varepsilon_t + F_1^{**}(L)\Delta\varepsilon_t]$$

$$= F_1(1)\,S_{2t} + F_1{}^*(1)S_{1t} + F_1^{**}(L)\varepsilon_t$$

$$\Rightarrow y_{1,t-2} = F_1(1)S_{2,t-2} + F_1^*(1)S_{1,t-2} + F_1^{**}(L)\varepsilon_{t-2}$$

- $\Delta y_{2t} = D_1(L)\Delta^2 y_{1t} + \theta_1 \Delta y_{1t} + v_{2t}$

$= D_1(L)F_1(L)\,\varepsilon_t + \theta_1[F_1(1)S_{1t} + F_1^*(L)\varepsilon_t] + G_2\,\varepsilon_t$
$= F_2(1)S_{1t} + \bar{G}_2(L)\,\varepsilon_t$
Using $F_2(1) = \theta_1 F_1(1),\; v_{2t} = G_2\,\varepsilon_t$

And $\bar{G}_2(L) = D_1(L)\,F_1(L) + \theta_1 f_1^*(L) + G_2$

Thus $y_{2t} = y_{2,t\ 1} + F_2(1)\,S_{1t} + \bar{G}_2(L)\varepsilon_t$ and recurrence yields

$y_{2t} = \sum_{s=1}^{t}[F_2(1)S_{1t} + \bar{G}_2(L)\varepsilon_t]$

$$= F_2(1)\sum_{s=1}^{t} S_{1t} + \sum_{s=1}^{t}[\bar{G}_2(1) + (1+L)\bar{G}_2^*(L)\,]\varepsilon_t$$

$$= F_2(1)S_{2t} + \bar{G}_2(1)\,S_{1t} + \bar{G}_2^*(L)\varepsilon_t$$

$$\Rightarrow y_{2,t-2} = F_2(1)\,S_{2,t-2} + \bar{G}_2(1)\,S_{1,t-2} + \bar{G}_2^*(L)\varepsilon_{t-2}$$

The two previous points lead to

$$\begin{pmatrix} y_{1,t-2} \\ y_{2,t-2} \end{pmatrix} = \begin{pmatrix} F_1^{**}(L) & F_1^*(1) & F_1(1) \\ \bar{G}_2^*\,(L) & \bar{G}_2^*\,(1) & F_2(1) \end{pmatrix} \begin{pmatrix} \varepsilon_{t-2} \\ S_{1,t-2} \\ S_{2,t-2} \end{pmatrix}$$

$$z_{2t} = K_1\,(L)\varepsilon_{t-2} + K_2\,(1)\,S_{1,t-2} + F_*\,(1)\,S_{2,t-2}$$

Where $K_1\,(L) = \begin{pmatrix} F_1^{**}(L) \\ \bar{G}_2^*\,(L) \end{pmatrix}$; $K_2 = \begin{pmatrix} F_1^*(1) \\ \bar{G}_2^*\,(1) \end{pmatrix}$; $F_*\,(1) = \begin{pmatrix} F_1(1) \\ F_2\,(1) \end{pmatrix}$

To sum up

1. $z_{0t} = K_0\ (L)\varepsilon_{t-1}$
2. $z_{1t} = F_1^*\,(L)\varepsilon_{t-1} + F_1\ (1)S_{1,t-1}$
3. $z_{2t} = K_1\,(L)\varepsilon_{t-2} + K_2\ (1)\,S_{1,t-2} + F_*\,(1)\,S_{2,t-2}$

We have

1. $T^{-1}\sum_{t=1}^{T} z_{0t}\ z'_{0t} \overset{P}{\rightarrow} E(z_{0t}\,z'_{0t})$

$$E(z_{0t}\ z'_{0t}) = E\,[K_0\ (L)\varepsilon_{t-1}\,\varepsilon'_{t-1}\,K_0\ (L)'] = K_0\,(L)\,K_0\,(L^{-1})'$$
$$= \sum_{j=0}^{\infty} k_{0j}\,k'_{0j} \equiv Q_{00}$$

2. $T^{-\frac{3}{2}}\sum_{t=1}^{T} z_{1t}\ z'_{0t} = T^{-\frac{3}{2}}\sum_{t=1}^{T}\left[F_1^*\,(L)\varepsilon_{t-1} + F_1\ (1)\,S_{1,t-1}\right][K_0(L)\,\varepsilon_{t-1}]'$

$$= T^{-\frac{1}{2}}\ T^{-1}\sum_{t=1}^{T}[F_1^*\,(L)\,\varepsilon_{t-1}\,][K_0\,(L)\varepsilon_{t-1}\,]'$$
$$+ T^{-\frac{1}{2}}\ F_1\,(1)\,T^{-1}\sum_{t=1}^{T} S_{1,t-1}\,[K_0(L)\varepsilon_{t-1}]'$$

$$\overset{L}{\rightarrow} 0\sum_{j=0}^{\infty} f_{1j}^*\ k'_{0j} + 0\,F_{11}\,(1)\left[\int_0^1 B\,(r)dB\,(r)'K'_0\ (1) + K'_0\ (1)\right] = 0$$

$$\Longrightarrow T^{-\frac{3}{2}}\sum_{t=1}^{T} z_{1t}\ z'_{0t} \overset{L}{\rightarrow} 0$$

Similary $T^{-\frac{3}{2}}\sum_{t=1}^{T} z_{0t}\ z'_{1t} = \left[T^{-\frac{3}{2}}\sum_{t=1}^{T} z_{1t}\ z'_{0t}\right]' \overset{L}{\rightarrow} 0$

3. $T^{-\frac{5}{2}} \sum_{t=1}^{T} \; z_{2t} \, z_{0t}' \; = \; T^{-\frac{5}{2}} \; \sum_{t=1}^{T} \left[K_1 \,(L)\varepsilon_{t-2} \; + \; K_2 \;(1)S_{1,t-2} \; + \; F_* \;(1)\, S_{2,t-2} \,\right][K_0\,(L)\varepsilon_{t-1}]'$

$$= T^{-\frac{3}{2}} \;\; T^{-1} \sum_{t=1}^{T} [K_1\,(L)\varepsilon_{t-2}][K_0\,(L)\varepsilon_{t-1}]'$$

$$+ \; T^{-\frac{3}{2}} \; K_2(1) \sum_{t=1}^{T} S_{1,t-2} \;\; [K_0(L)\varepsilon_{t-1}]'$$

$$+ \;\; T^{-\frac{1}{2}} F_* \;\; (1)T^{-2} \;\; \sum_{t=1}^{T} S_{2,t-2}[K_0(L)\varepsilon_{t-1}\,]'$$

$$\xrightarrow{L} 0 \sum_{j=0}^{\infty} k_{1j}, k_{0j}' \; + 0\left[\int_0^1 B(r)dB(r)'\,K_0\;(1)' + \; K_0 \;\;(1)'\right] + 0\left[\int_0^1 \bar{B} \;\;(r)dB(r)'K_0(1)'\right]$$

$$\Longrightarrow \; T^{-\frac{5}{2}} \; \sum_{t=1}^{T} z_{2t} \; z_{0t}' \; \xrightarrow{L} 0$$

Similary $T^{-\frac{5}{2}} \sum_{t=1}^{T} \, z_{0t} \; z_{2t}' \; = \left[T^{-\frac{5}{2}} \;\; \sum_{t=1}^{T} \, z_{2t} \; z_{0t}' \;\right]' \xrightarrow{L} 0$

4. $T^{\,-2} \;\; \sum_{t=1}^{T} \, z_{1t} \; z_{1t}' \; = \; T^{-2} \, \sum_{t=1}^{T} \left[F_1^* \,(L)\varepsilon_{t-1} \; + \; F_1 \;\,(1)S_{1,t-1}\right]\left[F_1^*(L)\varepsilon_{t-1} \, + F_1(1)S_{1,t-1}\right]'$

$$= \; T^{-1}\, T^{-1} \;\; \sum_{t=1}^{T} [F_1^* \;(L)\; \varepsilon_{t=1}\;][F_1^* \;(L)\varepsilon_{t-1}]'$$

$$+ \, T^{-1}T^{-1} \sum_{t=1}^{T} \;\; [F_1^* \;(L)\; \varepsilon_{t-1}]'\, S_{1,t-1}' \, F_1(1)'$$

$$+T^{-1}F_1 \, T^{-1} \sum_{t=1}^{T} S_{1,t-1} \;\; [F_1^* \;(L)\varepsilon_{t-1}]'$$

$$+ \;\; F_1 \;\; (1)T^{-2} \;\; \sum_{t=1}^{T} S_{1,t-1} \, S_{1,t-1}' \, F_1(1)'$$

As $T^{-1} \;\; \sum_{t=1}^{T} [F_1^* \;(L)\varepsilon_{t-1}][F_1^*(L)\varepsilon_{t-1}]'$; $T^{-1} \;\; \sum_{t=1}^{T} [F_1^* \;(L)\varepsilon_{t-1}]S_{1,t-1}']$

And $T^{-1} \;\sum_{t=1}^{T} \; S_{1,t-1}[F_1^* \,\varepsilon_{t-1}]'$ have finite limits and T^{-1}tends to zero, we have

$$T^{-2}\sum_{t=1}^{T} z_{1t}\, z_{1t}' \cong^{as} F_1(1)\; T^{-2}\sum_{t=1}^{T} S_{1,t-1}\, S_{1,t-1}' F_1(1)'$$

$$\overset{L}{\to} F_1(1)\int_0^1 B(r)B(r)'drF_1(1)'$$

$$\Longrightarrow T^{-2}\sum_{t=1}^{T} z_{1t}\, z_{1t}' \overset{L}{\to} F_1(1)\int_0^1 B(r)B(r)'drF_1(1)' \equiv Q_{11}$$

5. $T^{-3}\sum_{t=1}^{T} z_{2t}\, z_{1t}' = T^{-3}\sum_{t=1}^{T}\left[K_1(L)\varepsilon_{t-2} + K_2(1)S_{1,t-2} + F_*(1)S_{2,t-2}\right]\left[F_1^*(L)\varepsilon_{t-1} + F_1(1)S_{1,t-1}'\right]$

$$= T^{-2}T^{-1}\sum_{t=1}^{T}[K_1(L)\varepsilon_{t-2}][F_1^*(L)\varepsilon_{t-1}]'$$

$$+ T^{-2}\, T^{-1}\sum_{t=1}^{T}[K_1(L)\varepsilon_{t-2}]\, S_{1,t-1}' F_1(1)'$$

$$+ T^{-2}K_2(1)T^{-1}\sum_{t=1}^{T} S_{1,t-2}\,[F_1^*(L)\varepsilon_{t-1}]'$$

$$+ T^{-1}\, K_2(1)T^{-2}\sum_{t=1}^{T} S_{1,t-1}\, S_{1,t-1}'\, F_1(1)'$$

$$+ T^{-1}F_*(1)T^{-2}\sum_{t=1}^{T} S_{2,t-2}\,[F_1^*(L)\varepsilon_{t-1}]'$$

$$+ F_*(1)T^{-3}\sum_{t=1}^{T} S_{2,t-2}\, S_{1,t-1}'\, F_1(1)'$$

As before

$$T^{-3}\sum_{t=1}^{T} z_{2t}\, z_{1t}' \cong^{as} F_*(1)T^{-3}\sum_{t=1}^{T} S_{2,t-2}\, S_{1,t-1}\; F(1)'$$

$$\overset{L}{\to}\int_0^1 \bar{B}(r)B(r)'drF_1(1)'$$

$$\Longrightarrow T^{-3} \sum_{t=1}^{T} z_{2t}\, z_{1t}' \xrightarrow{L} F_* (1) \int_0^1 \bar{B} (r) B(r)' dr\, F_1 (1)' \equiv Q_{21}$$

Therefore,

$$T^{-3} \sum_{t=1}^{T} z_{1t}\, z_{2t}' \xrightarrow{L} F_1 (1) \int_0^1 B(r) \bar{B}(r)' dr F_*(1)' \equiv Q_{12}$$

6. $T^{-4} \sum_{t=1}^{T} z_{2t}\, z_{2t}' = T^{-4} \sum_{t=1}^{T} \left[K_1 (L)\varepsilon_{t-2} + K_2 (1) S_{1,t-2} + F_* (1) S_{2,t-2}\right]\left[K_1(L)\varepsilon_{t-2} + K_2(1) S_{1,t-2} + F_* (1)\, s_{2,t-2}\right]$

$$= T^{-4} \sum_{t=1}^{T} \left[k_1 (L)\varepsilon_{t-2} + K_2 (1) S_{1,t-2}\right]\left[K_1 (L)\varepsilon_{t-2} + K_2 (1) S_{1,t-2} + F_* (1) S_{2,t-2}\right]'$$

$$+ T^{-4} \sum_{t=1}^{t} \left[F_* (1)\, S_{2,t-2}\right]\left[K_1(L)\,\varepsilon_{t-2} + K_2 (1) S_{1,t-2}'\right]$$

$$+ T^{-4} \sum_{t=1}^{T} \left[F_* (1)\, S_{2,t-2}\right]\left[F_* (1) S_{2,t-2}\right]'$$

In the same way as for point 4. above,

$$T^{-4} \sum_{t=1}^{T} z_{2t}\, z_{2t}' \cong^{as} F_* (1)\, T^{-4} \sum_{t=1}^{T} S_{2,t-2}\, S_{2,t-2}'\, F_* (1)' \equiv Q_{22}$$

$$T^{-1} \sum_{t=1}^{T} z_{1t} \otimes w_{3t} = \frac{1}{T} \sum_{t=1}^{T} \left[F_1^{**} (L)\varepsilon_{t-1} + F_1 (1) S_{1,t-1}\right] \otimes w_{3t}$$

$$= T^{-1} \sum_{t=1}^{T} F_1^{*} (L)\, \varepsilon_{t-1} \otimes w_{3t} + T^{-1} \sum_{t=1}^{T} F_1 (1)\, S_{1n\,t-1} \otimes w_{3t}$$

$$= vec\left[T^{-1}\sum_{t=1}^{t} w_{3t}\,(F_1^*\,(L)\varepsilon_{t-1})'\right] + vec\left[T^{-1}\sum_{t=1}^{t} w_{3t}\,(F_1^*\,(L)S_{t-1})'\right]$$

As w_{3t} and $\varepsilon_{t-s}\,, s = 1, 2, 3, \dots$ Are independent, we have

- $T^{-1}\sum_{t=1}^{T} w_{3t}\,(F_1^*\,(L)\varepsilon_{t-1})' \overset{L}{\to} 0$
- $T^{-1}\sum_{t=1}^{T} w_{3t}\left(F_1\,(1)S_{1,t-1}\right)' \overset{L}{\to} \int_0^1 d\,B_{03}\,(r)'\,B\,(r)'\;F_1\,(1)'$

$$\Rightarrow T^{-1}\sum_{t=1}^{T} z_{1t}\otimes w_{3t} \overset{L}{\to} vec\,[\int_0^1 d\,B_{03}\,(r)\otimes B_{03}\,(r)'F_1(1)]$$

$$= [F_1\,(1)\otimes I_{n3}]\int_0^1 B\,(r)\otimes\,dB_{03}\,(r) \equiv \varphi_{1*}$$

$$T^{-2}\sum_{t=1}^{T} z_{2t}\otimes w_{3t}$$

$$= T^{-2}\sum_{t=1}^{T}\big[K_1\,(L)\varepsilon_{t-2} + K_2\,(1)\,S_{1,t-1} + F_*\,(1)S_{2;t-2}\big]\otimes w_{3t}$$

$$T^{-2}\sum_{t=1}^{T} K_1\,(L)\varepsilon_{t-2}\otimes w_{3t} + T^{-2}\sum_{t=1}^{T} k_1\,(1)S_{1,t-2}\otimes w_{2t} + T^{-2}\sum_{t=1}^{T} F_*\,(1)S_{2,t-2}\otimes w_{3t}$$

With T^{-1}tending to zero, the last two terms above have zero limits. Therefore

$$T^{-2}\sum_{t=1}^{T} z_{2t}\ \otimes w_{3t}\ \cong^{as}\ T^{-2}\sum_{t=1}^{T} F_{*}\,(1)S_{2,t-2}\otimes\ w_{3t}$$

$$= vec\left[T^{-2}\sum_{t=1}^{T} w_{3t}\ (L)\big(F_{*}\ (1)S_{2,t-2}\big)'\right]$$

$$= vec\ [\int_{0}^{1} d\ B_{03}\ (r)\bar{B}\ (r)'F_{*}\ (1)']$$

$$= \int_{0}^{1} F_{*}\ (1)\bar{B}(r)\otimes\ d\ B_{03}\ (r)$$

$$\Rightarrow\ T^{-2}\sum_{t=1}^{T} z_{2t}\otimes w_{3t}\ \overset{L}{\rightarrow}\ [F_{*}\,(1)\otimes\ I_{n3}\,]\int_{0}^{1}\bar{B}\ (r)\otimes dB_{03}(r)\equiv\ \varphi_{2*}$$

Thus

$$\varphi_{T}\ =\sum_{t=1}^{T}\Gamma_{T}^{-1}\, z_{t}\otimes\ w_{3t}\ \overset{L}{\rightarrow}\begin{pmatrix}\varphi_{0*}\\ \varphi_{1*}\\ \varphi_{2*}\end{pmatrix}$$

From all these results, we can conclude that

$$\left(\Gamma_{T}\otimes I_{n_3}\right)\left(vec\,\widehat{\Theta}_{ols}\ - vec\ \Theta\ \right)\overset{L}{\rightarrow}\ (Q\ \ \otimes I_{n3}\,)^{-1}\,\varphi_{*}$$

Thus

$$T^{\frac{1}{2}}\big(vec\Theta_{0,ols} - vec\Theta_{0}\big)\overset{L}{\rightarrow}\big(Q_{00}^{-1}\otimes I_{n_3}\big)\varphi_{0*}\sim N[0,Q_{00}^{-1}\otimes\Omega_{3}]\quad (A4.1)$$

$$\begin{pmatrix}T\big(vec\widehat{\Theta}_{1\,ols}\ - vec\Theta_{1}\big)\\ T^{2}\ \big(vec\widehat{\Theta}_{2\,ols}\ - vec\Theta_{2}\big)\end{pmatrix}\overset{L}{\rightarrow}\left[\begin{pmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}\otimes I_{n_3}\right]\begin{pmatrix}\varphi_{1*}\\ \varphi_{2*}\end{pmatrix}\qquad (A4.2)$$

Knowing

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}^{-1} = \begin{pmatrix} (Q_{11} - Q_{12}\, Q_{22}^{-1} Q_{21})^{-1} & -(Q_{11} - Q_{12}\, Q_{22}^{-1}\; Q_{12}\, Q_{22}^{-1}) \\ -(Q_{22} - Q_{21}\; Q_{11}^{-1}\, Q_{12})^{-1}\, Q_{21}\, Q_{11}^{-1} & (Q_{22} - Q_{21}\, Q_{11}^{-1}\, Q_{12})^{-1} \end{pmatrix}$$

We have

$$T\left(vec\, \widehat{\Theta}_{1\,ols} - vec\; \Theta_1\right) \xrightarrow{L} [(Q_{11} - Q_{12}\, Q_{22}^{-1} Q_{21})^{-1} \otimes I_{n3}][\varphi_{1*} - (Q_{12}\, Q_{22}^{-1} \otimes I_{n3})\; \varphi_{2*}]\; (A4.3)$$

$$T^2\left(vec\, \widehat{\Theta}_{2\,ols} - vec\; \Theta_2\right) \xrightarrow{L} [(Q_{22} - Q_{21}\; Q_{11}^{-1}\, Q_{12})^{-1} \otimes I_{n3}][\varphi_{2*\,-}(Q_{21}\; Q_{11}^{-1} \otimes I_{n3})\varphi_{1*}\,]\; (A4.4)$$